Lecture on

New approaches to nonlinear control of robotic systems:

Global linearization methods

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1. Outline

- It is proven that the nonlinear model of the **underactuated vessel** is a **differentially flat** one. It is shown that this model cannot be subjected to static feedback linearization, however it admits **dynamic feedback linearization** which means that the **system's state vector is extended** by including as additional state variables the control inputs and their derivatives.

- Next, using the **differential flatness properties** it is also proven that this model can be subjected to **input-output linearization** and can be transformed to an equivalent **canonical (Brunovsky) form**. Based on this the design of a state **feedback controller** is carried out enabling accurate manoeuvring and trajectory tracking.

- The **Derivative-free nonlinear Kalman Filter** is used as **disturbance observer** for dynamically identifying model uncertainty and external perturbation terms. .

- This nonlinear filter consists of the Kalman Filter's recursion on the linearized equivalent model of the vessel and of an inverse **nonlinear transformation based on the differential flatness** features of the system which enables to compute state estimates for the state variables of the initial nonlinear model.

- The redesign of the filter as a **disturbance observer** makes possible the estimation and **compensation** of additive **perturbation terms** affecting the vessel's model.

2. Model of the underactuated vessel

- The underactuated vessel's model stems from the **generic ship's model**, after setting specific values for the elements of the inertia and Coriolis matrix and after reducing the number of the available control inputs.

- The **state-space equation** of the nonlinear underactuated vessel is

$$
\begin{aligned}
\dot{x} &= u\cos(\psi) - v\sin(\psi) \\
\dot{y} &= u\sin(\psi) + v\cos(\psi) \\
\dot{\psi} &= r \\
\dot{u} &= v{\cdot}r + \tau_u \\
\dot{v} &= -u{\cdot}r - \beta v \\
\dot{r} &= \tau_r
\end{aligned}
$$

x and *y* are the cartesian coordinates of the vessel

ψ is the orientation angle

u is the surge velocity

v is the sway velocity

r is the yaw rate

The **control inputs** are the surge force τ_u and the yaw torque τ_r

The underactuated vessel's model is also written in the **matrix form**

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} u\cos(\psi) - v\sin(\psi) \\ u\sin(\psi) + v\cos(\psi) \\ r \\ vr \\ -ur - \beta v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_u \\ \tau_r \end{pmatrix}
$$

2. Model of the underactuated vessel

or equivalently, one has the description $\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{v}$

The system's state vector is denoted as $\tilde{x} = [x, y, \psi, u, v, r]^T$

while $f(\tilde{x}) \in R^{6\times 1}$ and $\tilde{g}(\tilde{x}) = [\tilde{g}_a, \tilde{g}_b] \in R^{6\times 2}$

while the control input is the vector $\tilde{v} = [\tau_u, \tau_r]^T$

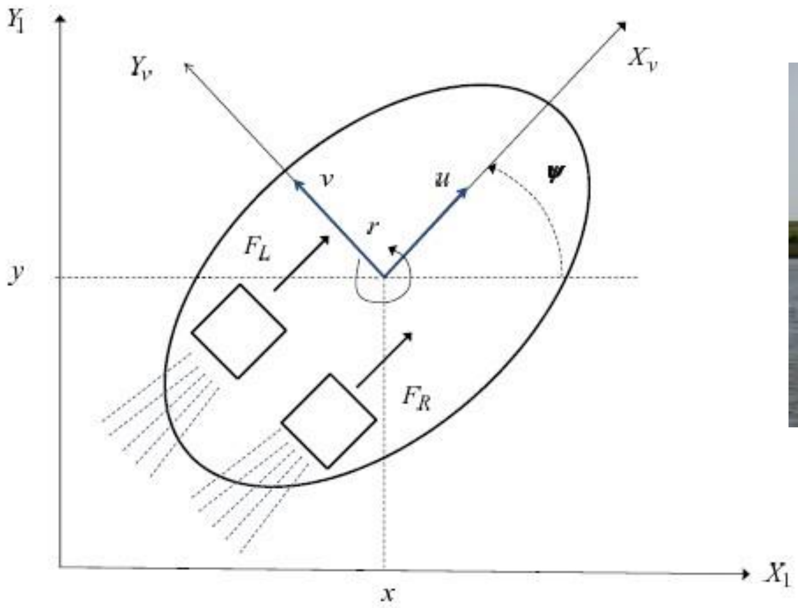

Fig. 1. Diagram of the underactuated hovercraft's kinematic model

2. Model of the underactuated vessel

The system's state vector can be extended by including as additional state variables the control input τ_u and its first derivative $\dot{\tau}_u$.

The extended state-space description of the system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} ucos(\psi) - vsin(\psi) \\ usin(\psi) + vcos(\psi) \\ r \\ vr + z_1 \\ -ur - \beta v \\ 0 \\ z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \end{pmatrix}$$

or equivalently, one has the description $\dot{z} = f(z) + g(z)\tilde{v}$

The extended system's state vector is denoted as $z = [x, y, \psi, u, v, r, z_1, z_2]^T$

Moreover, one has $f(z) \in R^{8\times 1}$ and $g(z) = [g_a, g_b] \in R^{8\times 2}$,

while the control input is the vector is $\tilde{v} = [\ddot{\tau}_u, \tau_r]^T$.

3. Outline of differential flatness theory

- **Differential flatness theory** has been developed as a **global linearization control method** by M. Fliess (Ecole Polytechnique, France) and co-researchers (Lévine, Rouchon, Mounier, Rudolph, Petit, Martin, Zhu, Sira-Ramirez et. al)

- A dynamical system can be written in the ODE form $S_i(w, \dot{w}, \ddot{w}, \ldots, w^{(i)}),\ \ i = 1, 2, \ldots, q$

  where $w^{(i)}$ stands for the i-th derivative of either a state vector element or of a control input

- The system is said to be **differentially flat** with respect to the **flat output**

$$y_i = \phi(w, \dot{w}, \ddot{w}, \ldots, w^{(a)}),\ i = 1, \ldots, m \quad \text{where} \quad y = (y_1, y_2, \ldots, y_m)$$

if the following two conditions are satisfied

(i) There does not exist any differential relation of the form

$$R(y, \dot{y}, \ddot{y}, \ldots, y^{(\beta)}) = 0$$

which means that **the flat output and its derivatives are linearly independent**

(ii) All system variables are **functions of the flat output and its derivatives**

$$w^{(i)} = \psi(y, \dot{y}, \ddot{y}, \ldots, y^{(\gamma_i)})$$

3. Outline of differential flatness theory

The proposed Lyapunov theory-based control method is based on the **transformation** of the nonlinear system's model into the **linear canonical form**, and this transformation is succeeded by exploiting the system's differential flatness properties

- **All single input nonlinear systems** are differentially flat and can be transformed into the linear canonical form

One has to define also which are the **MIMO nonlinear systems** which are differentially flat.

- Differential flatness holds for **MIMO nonlinear systems** that admit **static feedback linearization.**and which can be transformed into the linear canonical form through a change of variables (diffeomorphism) and feedback of the state vector.

- Differential flatness holds for **MIMO nonlinear models** that admit **dynamic feedback linearization**, This **is the case of specific underactuated robotic models.** In the latter case the state vector of the system is extended by considering as additional flat outputs some of the control inputs and their derivatives

- Finally, a more rare case is the so-called **Liouvillian systems**. These are systems for which differential flatness properties hold for part of their state vector (constituting a flat subsystem) while the non-flat state variables can be obtained by integration of the elements of the flat subsystem.

4. Differential flatness of the model of the underactuated vessel

The flat output is the vector of the vessel's cartesian coordinates, that is

$$\tilde{y} = [y_1, y_2] = [x, y]$$

It holds that

$$\begin{aligned} \ddot{x} &= \dot{u}cos(\psi) - u{\cdot}sin(\psi){\cdot}\dot{\psi} - \dot{v}sin(\psi) - v{\cdot}cos(\psi)\dot{\psi} \\ \ddot{y} &= \dot{u}sin(\psi) + u{\cdot}cos(\psi){\cdot}\dot{\psi} + \dot{v}cos(\psi) - v{\cdot}cos(\psi)\dot{\psi} \end{aligned} \qquad 1$$

Moreover, it holds that

$$\begin{aligned} \ddot{x} + \beta\dot{x} &= cos(\psi)(\dot{u} - v\dot{\psi} + \beta u) + sin(\psi)(-u\dot{\psi} - \dot{v} - \beta v) \\ \ddot{y} + \beta\dot{y} &= cos(\psi)(\dot{v} + u\dot{\psi} + \beta v) + sin(\psi)(-v\dot{\psi} + \dot{u} + \beta u) \end{aligned} \qquad 2$$

Using Eq. 1 and Eq. 2 , and after computing that

$$\begin{aligned} u\dot{\psi} + \dot{v} + \beta v &= u{\cdot}r - ur - \beta v + \beta v = 0 \\ \dot{u} - v\dot{\psi} + \beta u &= vr + \tau_u - vr + \beta u = \tau_u + \beta u \end{aligned}$$

one obtains that

$$\frac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}} = \frac{cos(\psi)0+sin(\psi)(\tau_u+\beta u)}{cos(\psi)(\tau_u+\beta u)-sin(\psi)0} \Rightarrow$$

$$\frac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}} = tan(\psi) \Rightarrow \psi = tan^{-1}\left(\frac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}}\right) \qquad 3$$

4. Differential flatness of the model of the underactuated vessel

Through Eq. 3 it is proven that the state variable ψ (heading angle of the vessel) is a function of the flat output and of its derivatives.

From Eq. 1 one also has that

$$(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2=(\tau_u+\beta u)^2 \quad (4)$$

Moreover, it holds that

$$\begin{aligned}\dot{x}(\ddot{x}+\beta\dot{x})&=(ucos(\psi)-vsin(\psi))cos(\psi)(\tau_u+\beta u)\\ \dot{y}(\ddot{y}+\beta\dot{y})&=(usin(\psi)+vcos(\psi))sin(\psi)(\tau_u+\beta u)\end{aligned} \quad (5)$$

while using Eq. 4 and after intermediate computations one finally obtains

$$\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})=u(\tau_u+\beta u) \quad (6)$$

which finally gives

$$u=\frac{\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}} \quad (7)$$

It also holds that

4. Differential flatness of the model of the underactuated vessel

$$\dot{y}\ddot{x}-\dot{x}\ddot{y} \;=\; (usin(\psi)+vcos(\psi))(\dot{u}cos(\psi)-usin(\psi)\dot{\psi}-\dot{v}sin(\psi)-vcos(\psi)\dot{\psi})-(ucos(\psi)-vsin(\psi))(\dot{u}sin(\psi)+ucos(\psi)\dot{\psi}+\dot{v}cos(\psi)-vsin(\psi)\dot{\psi})$$

which after intermediate computations and substitution of the derivative variables gives

$$\dot{y}\ddot{x}-\dot{x}\ddot{y}=v(\beta u+\tau_u) \qquad (8)$$

From Eq. 8 and Eq. 4 one gets

$$v=\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\sqrt{(\ddot{x}+\beta\dot{x})^2+\ddot{y}+\beta\dot{y})^2}} \qquad (9)$$

From the state-space equations it holds that

$$r=\dot{\psi} \qquad (10)$$

and using Eq. 10 one also has that r is a differential function of the flat output

4. Differential flatness of the model of the underactuated vessel

This can be also confirmed analytically. Indeed from Eq (3) it holds

$$\frac{cos^2(\psi)\dot{\psi}+sin^2(\psi)\dot{\psi}}{cos^2(\psi)} = \frac{(y^{(3)}+\beta\ddot{\psi})(\ddot{x}+\beta\dot{x})-(\ddot{y}+\beta\dot{y})(x^{(3)}+\beta\ddot{x})}{(\ddot{x}+\beta\dot{x})^2} \quad (11)$$

which also gives

$$\frac{\dot{\psi}}{cos^2(\psi)} = \frac{(y^{(3)}+\beta\ddot{\psi})(\ddot{x}+\beta\dot{x})-(\ddot{y}+\beta\dot{y})(x^{(3)}+\beta\ddot{x})}{(\ddot{x}+\beta\dot{x})^2} \quad (12)$$

while also using that

$$\frac{1}{\cos^2\psi} = tan^2(\psi)+1 \quad (13)$$

one obtains that

$$\cos^2\psi = \frac{(\ddot{x}+\beta\dot{x})^2}{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2} \quad (14)$$

Thus, from Eq. (12) and Eq. (10) one has

$$r=\dot{\psi} \Rightarrow r = cos^2(\psi)\frac{(y^{(3)}+\beta\ddot{\psi})(\ddot{x}+\beta\dot{x})-(\ddot{y}+\beta\dot{y})(x^{(3)}+\beta\ddot{x})}{(\ddot{x}+\beta\dot{x})^2} \quad (15)$$

Equivalently, from the extended state-space equations of the system one has that

4. Differential flatness of the model of the underactuated vessel

$$\tau_u = \dot{u} - v \cdot r \Rightarrow \tau_u = \frac{d}{dt}\left\{\frac{\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}\right\} -$$

$$-\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}} \cdot \frac{y^{(3)}(\ddot{x}+\beta\dot{x})-x^{(3)}(\ddot{y}+\beta\dot{y})-\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})}{(\ddot{x}+\beta\dot{x})^2+\ddot{y}+\beta\dot{y})^2} \quad (16)$$

which after intermediate operations gives $\tau_u = \frac{\ddot{x}(\ddot{x}+\beta\dot{x})+\ddot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}$ (17)

Finally, using that the control input $\tau_r = \dot{r}$ this implies also that τ_r is a differential function of the flat output

The above can be also shown analytically

$$\tau_r = \dot{r} \Rightarrow \tau_r =$$

$$\frac{y^{(4)}(\ddot{x}+\beta x)-x^{(4)}(\ddot{y}+\beta\dot{y})+\beta(y^{(3)}\ddot{x}-x^{(3)}\ddot{y})-\beta^2(x^{(3)}\dot{y}-y^{(3)}\dot{x})}{[(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2]\cdot}$$
$$-2\frac{[y^{(3)}(\ddot{x}+\beta\dot{x})-x^3(\ddot{y}+\beta\dot{y})-\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})]}{[(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2]^2}\cdot$$
$$\cdot\{(\ddot{x}+\beta\dot{x})(x^{(3)}+\beta\ddot{x})+(\ddot{y}+\beta\dot{y})(y^{(3)}+\beta\ddot{y})\} \quad (18)$$

Thus it is confirmed that the model of the underactuated vessel is a differentially flat one

5. Flatness-based control of the underactuated vessel

Next, it will be shown that a flatness-based controller can be developed for the model of the underactuated vessel. It has been shown that it holds

$$\ddot{x} = \dot{u}cos(\psi) - usin(\psi)\dot{\psi} - \dot{v}sin(\psi) - vcos(\psi)\dot{\psi} \Rightarrow \ddot{x} = (vr + \tau_u)cos(\psi) - usin(\psi)r - (-ur - \beta v)sin(\psi) - vcos(\psi)r \Rightarrow \ddot{x} = \tau_u cos(\psi) + \beta vsin(\psi)$$

By differentiating once more with respect to time and after intermediate operations one finally obtains

$$x^{(3)} = \dot{\tau}_u cos(\psi) - \tau_u sin(\psi)r + \\ +\beta(-ur - \beta v)sin(\psi) + \beta vcos(\psi)r \quad (19)$$

Similarly one has

$$\ddot{y} = \dot{u}sin(\psi) + ucos(\psi)\dot{\psi} + \dot{v}cos(\psi) - vsin(\psi)\dot{\psi} \Rightarrow \ddot{y} = (vr + \tau_u)sin(\psi) + ucos(\psi)r + (-ur - \beta v)cos(\psi) - vsin(\psi)r \Rightarrow \ddot{y} = \tau_u sin(\psi) - \beta vcos(\psi)$$

By differentiating once more with respect to time and by using the state variables of the extended state-space model $z_1 = \tau_u$ and $z_2 = \dot{\tau}_u$ one finally obtains

$$y^{(3)} = z_2 sin(\psi) + z_1 cos(\psi)r + \beta urcos(\psi) + \\ +\beta^2 vcos(\psi) + \beta vsin(\psi)r \quad (20)$$

5. Flatness-based control of the underactuated vessel

Eq. 19 Is differentiated once again with respect to time, so as the control input τ_r to appear

$$x^{(4)} = [-2z_2 sin(\psi)r - z_1 cos(\psi)r^2 - \beta v r^2 sin(\psi) - \beta z_1 r sin(\psi) - \beta u r^2 cos(\psi) + \beta^2 u r sin(\psi) - \beta^3 v sin(\psi) - \beta^2 v r cos(\psi) - \beta u r^2 cos(\psi) + \beta^2 v r cos(\psi) - \beta v r^2 sin\psi] + [cos(\psi)]\ddot{\tau}_u + [-z_1 sin(\psi) - \beta u sin(\psi) + \beta v cos(\psi)]\tau_r \quad (21)$$

Using a Lie algebra-based formulation Eq. 22 Is written in the form

$$x^{(4)} = L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r \quad (22)$$

where

$$L_f^4 y_1 = -2z_2 sin(\psi)r - z_1 cos(\psi)r^2 - \beta v r^2 sin(\psi) - \beta z_1 r sin(\psi) - \beta u r^2 cos(\psi) + \beta^2 u r sin(\psi) - \beta^3 v sin(\psi) - \beta^2 v r cos(\psi) - \beta u r^2 cos(\psi) + \beta^2 v r cos(\psi) - \beta v r^2 sin\psi$$

$$L_{g_a} L_f^3 y_1 = cos(\psi)$$

$$L_{g_b} L_f^3 y_1 = -z_1 sin(\psi) - \beta u sin(\psi) + \beta v cos(\psi)$$

Eq. 20 Is differentiated once again with respect to time, so as the control input $\ddot{\tau}_\theta$ to appear

5. Flatness-based control of the underactuated vessel

This gives

$$y^{(4)} = [z_2 rcos(\psi) - z_1 r^2 sin(\psi) + \beta ur^2 sin(\psi) + \beta^2 vrsin(\psi) - \beta vr^2 cos(\psi)] - \beta vr^2 cos(\psi) - \beta z_1 rcos(\psi) + \beta ur^2 sin(\psi) - \beta urcos(\psi) + \beta^2 vcos(\psi) - \beta^2 vrsin(\psi) + z_2 rcos(\psi)] + [sin(\psi)]\ddot{\tau}_u + [z_1 cos(\psi) - \beta vsin(\psi) - \beta ucos(\psi)]\tau_r$$

which after using a Lie algebra-based formulation is written as

$$y^{(4)} = L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r \quad (23)$$

where $L_f^4 y_2 = [z_2 rcos(\psi) - z_1 r^2 sin(\psi) + \beta ur^2 sin(\psi) - \beta^2 vrsin(\psi) - \beta vr^2 cos(\psi)] - \beta vr^2 cos(\psi) - \beta z_1 rcos(\psi) + \beta ur^2 sin(\psi) - \beta urcos(\psi) + \beta^2 vcos(\psi) - \beta^2 vrsin(\psi) + z_2 rcos(\psi)]$, and

$$L_{g_a} L_f^3 y_2 = sin(\psi)$$

$$L_{g_b} L_f^3 y_2 = z_1 cos(\psi) - \beta vsin(\psi) - \beta ucos(\psi)$$

5. Flatness-based control of the underactuated vessel

Consequently, the aggregate input-output linearized description of the system becomes

$$
\begin{aligned}
x^{(4)} &= L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r \\
y^{(4)} &= L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r
\end{aligned}
$$

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while by defining the new control inputs

$$
\begin{aligned}
v_1 &= L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r \\
v_2 &= L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r
\end{aligned}
$$

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one gets

$$
\begin{aligned}
x^{(4)} &= v_1 \\
y^{(4)} &= v_2
\end{aligned}
$$

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For the dynamics of the linearized equivalent model of the system the following new state variables can be defined

$$
\begin{aligned}
&z_{1,1} = x \quad && z_{1,2} = \dot{x} \quad && z_{1,3} = \ddot{x} \quad && z_{1,4} = x^{(3)} \\
&z_{2,1} = y \quad && z_{2,2} = \dot{y} \quad && z_{2,3} = \ddot{y} \quad && z_{2,4} = y^{(3)}
\end{aligned}
$$

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5. Flatness-based control of the underactuated vessel

and the state-space description of the system becomes

$$\dot{z} = Az + Bv$$
$$z^m = Cz \qquad (28)$$

or equivalently

$$\begin{pmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \\ \dot{z}_{1,4} \\ \dot{z}_{2,1} \\ \dot{z}_{2,2} \\ \dot{z}_{2,3} \\ \dot{z}_{2,4} \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \\ z_{1,4} \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ z_{2,4} \end{pmatrix} + \begin{pmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 1&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad (29)$$

while the associated measureme

$$\begin{pmatrix} z_1^m \\ z_2^m \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \\ z_{1,4} \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ z_{2,4} \end{pmatrix} \qquad (30)$$

5. Flatness-based control of the underactuated vessel

A suitable feedback control law for the linearized system is

$$v_1 = x_d^{(4)} - k_1^1(x^{(3)} - x_d^{(3)}) - k_2^1(\ddot{x} - \ddot{x}_d) - k_3^1(\dot{x} - \dot{x}_d) - k_4^1(x - x_d), \text{ and } v_2 = y_d^{(4)} - k_1^2(y^{(3)} - y_d^{(3)}) - k_2^2(\ddot{y} - \ddot{y}_d) - k_3^2(\dot{y} - \dot{y}_d) - k_4^2(y - y_d) \quad (31)$$

One can compute again the control input that is finally applied to the model of the underactuated vessel. It holds that

$$\bar{v} = \tilde{f} + \tilde{M}\tilde{v} \quad (32)$$

where the following matrices and vectors are defined:

$$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \tilde{f} = \begin{pmatrix} L_f^4 z_{1,1} \\ L_f^4 z_{2,1} \end{pmatrix} \quad (33)$$

$$\tilde{M} = \begin{pmatrix} L_{g,a}L_f^3 z_{1,1} & L_{g_b}L_f^3 z_{1,1} \\ L_{g,a}L_f^3 z_{2,1} & L_{g,b}L_f^3 z_{2,1} \end{pmatrix} \qquad \tilde{v} = \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \end{pmatrix}$$

The stabilizing control input that is finally exerted on the vessel is

$$\tilde{v} = \tilde{M}^{-1}(\bar{v} - \tilde{f}) \quad (34)$$

5. Flatness-based control of the underactuated vessel

For the linearized equivalent model of the system it is possible to perform state estimation using the Derivative-free nonlinear Kalman Filter.

Before computing the Kalman Filter stages, the previously defined matrices A,B and C are substituted by their discrete-time equivalentsA_d, B_d and C_d.

This is done through common discretization methods. The recursion of the filter's algorithm consists of two stages:

Measurement update::

$$\begin{aligned} K(k) &= P^- C_d^T [P^- C_d^T P + R]^{-1} \\ \hat{z}(k) &= \hat{z}^-(k) - K(k)[C_d z(k) - C_d \hat{z}^-(k)] \\ P(k) &= P^-(k) - K(k) C_d P^-(k) \end{aligned} \quad (36)$$

Time update::

$$\begin{aligned} P^-(k+1) &= A_d^T P(k) A_d + Q(k) \\ \hat{z}^-(k+1) &= A_d \hat{z}(k) + B_d u(k) \end{aligned} \quad (37)$$

Moreover, using the inverse transformations described by Eq. 3 7 9 10

one obtains estimates for the state variables of the initial nonlinear system.

6. Disturbances compensation with the use of Kalman Filtering

It is assumed that the input-output linearized equivalent model of the system, is subjected to disturbance terms which express the effects of both modelling uncertainty and of external perturbations. Thus one has

$$x^{(4)} = v_1 + \tilde{d}_1$$
$$y^{(4)} = v_2 + \tilde{d}_2 \quad (38)$$

It is considered that the disturbance signals are equivalently represented by their time derivatives (up to order n) and by the associated initial conditions (however, since disturbances are estimated with the use of the Kalman Filter, finally the dependence on knowledge of initial conditions becomes obsolete). It holds that

$$\tilde{d}_1^{(n)} = f_{d_1} \quad \tilde{d}_2^{(n)} = f_{d_2} \quad (39)$$

The state vector of the system is extended to include as additional tate variables the disturbance inputs and their derivatives. Thus one obtains

$$z_{d,1} = \tilde{d}_1 \quad z_{d,2} = \dot{\tilde{d}}_1 \quad z_{d,3} = \tilde{d}_2 \quad z_{d,4} = \dot{\tilde{d}}_2 \quad (40)$$

Thus, the extended state-space description of the system becomes:

6. Disturbances compensation with the use of Kalman Filtering

$$\begin{pmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \\ \dot{z}_{1,4} \\ \dot{z}_{2,1} \\ \dot{z}_{2,2} \\ \dot{z}_{2,3} \\ \dot{z}_{2,4} \\ \dot{z}_{d,1} \\ \dot{z}_{d,2} \\ \dot{z}_{d,3} \\ \dot{z}_{d,4} \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \\ z_{1,4} \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ z_{2,4} \\ z_{d,1} \\ z_{d,2} \\ z_{d,3} \\ z_{d,4} \end{pmatrix} + \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \\ f_{d_1} \\ f_{d_2} \end{pmatrix} \tag{41}$$

and the measurement equation becomes

$$\begin{pmatrix} z_{1,1} \\ z_{2,1} \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \end{pmatrix} z_e \tag{42}$$

where $z_e = [z_{1,1}, z_{1,2}, z_{1,3}, z_{1,4}, z_{2,1}, z_{2,2}, z_{2,3}, z_{2,4}, z_{d,1}, z_{d,2}, z_{d,3}, z_{d,4}]^T$

Thus, the extended state-space description of the system becomes:

6. Disturbances compensation with the use of Kalman Filtering

$$\dot{z}_e = A_e z_e + B_e v_e$$
$$z_e^{meas} = C_e z_e \quad (43)$$

For the extended state-space description of the system one can design a state estimator of the form

$$\dot{\hat{z}}_e = A_o z_e + B_o v_e + K(z_e^{meas} - C_o \hat{z}_e)$$
$$\hat{z}_e^{meas} = C_o \hat{z}_e \quad (44)$$

where for matrices A_o and C_o it holds $A_o = A$ and $C_o = C$ while for matrix B_o it holds

$$B_o^T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Again the Kalman Filter recursion provides joint estimation of the non-measurable state vector elements, of the disturbances' inputs and of their derivatives.

Prior to computing the Kalman Filter stages, the previously defined matrices A,B and C are substituted by their discrete-time equivalents A_{ed} , B_{ed} and C_{ed}.

6. Disturbances compensation with the use of Kalman Filtering

The recursion of the filter's algorithm consists of two stages. Thus, one has

Measurement update::

$$K(k) = P_e^- C_{e_d}^T [P_e^- C_{e_d}{}^T P_e + R_e]^{-1}$$
$$\hat{z}_e(k) = \hat{z}_e^-(k) - K(k)[C_{e_d} z_e(k) - C_{e_d} \hat{z} e^-(k)]$$
$$P_e(k) = P_e^-(k) - K(k) C_{e_d} P_e^-(k) \quad (45)$$

Time update::

$$P_e^-(k+1) = A_{e_d}{}^T P_e(k) A_{e_d} + Q_e(k)$$
$$\hat{z}_e^-(k+1) = A_{e_d} \hat{z}_e(k) + B_{e_d} v_e(k) \quad (46)$$

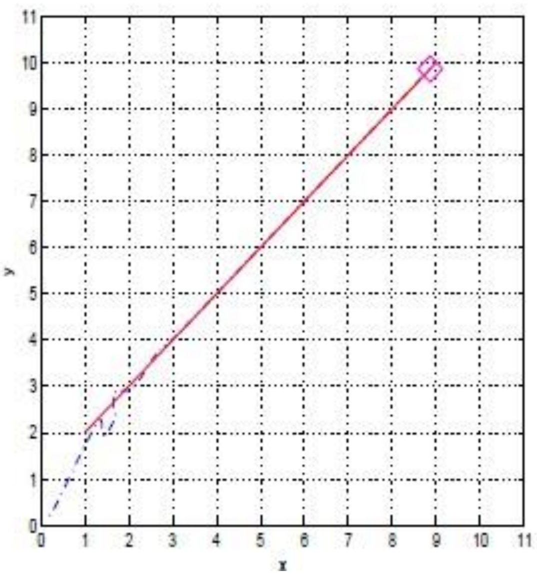
For compensating the disturbances effects, the modified control input that is applied to the system is

$$v_1 = x_d^{(4)} - k_1^1(x^{(3)} - x_d^{(3)}) - k_2^1(\ddot{x} - \ddot{x}_d) - k_3^1(\dot{x} - \dot{x}_d) - k_4^1(x - x_d) - \hat{z}_{d,1} \text{ and } v_2 = y_d^{(4)} - k_1^2(y^{(3)} - y_d^{(3)}) - k_2^2(\ddot{x} - \ddot{y}_d) - k_3^2(\dot{y} - \dot{y}_d) - k_4^2(y - y_d) - \hat{z}_{d,2}. \quad (47)$$

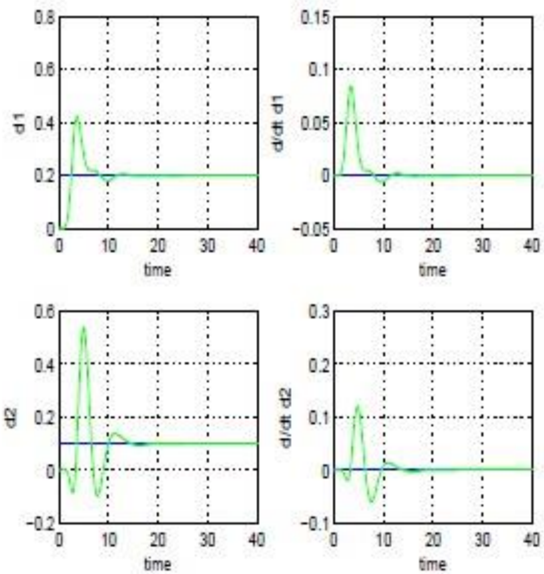
7. Simulation tests

In simulation tests It has been observed that in all cases the nonlinear feedback controller succeeded fast and accurate tracking of the reference setpoints.

The Derivative-free nonlinear Kalman Filter enabled estimation of the non-measurable variables of the system's state-vector which were needed for the implementation of the feedback control scheme

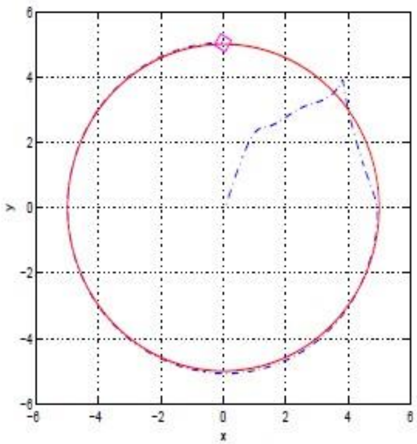


Reference path 1: Trajectory tracking for states x,y of the underactuated hovercraft

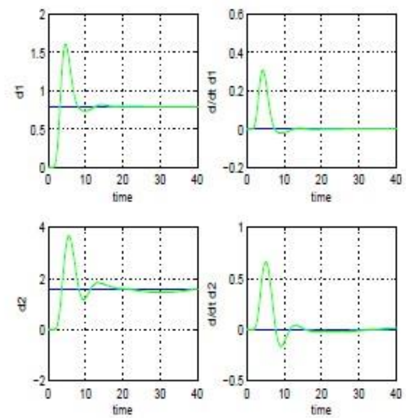


Reference path 1: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter

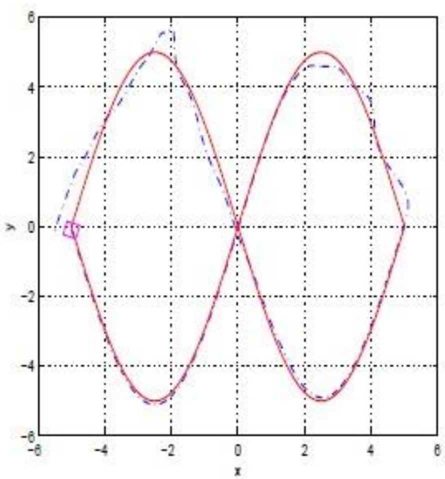
7. Simulation tests



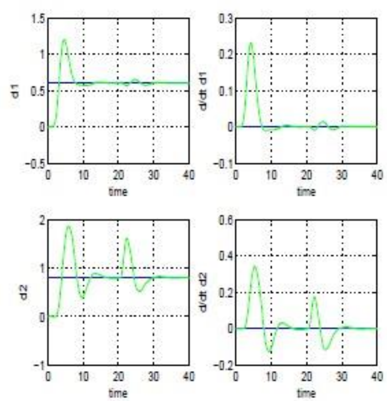
Reference path 2: Trajectory tracking for states x,y of the underactuated hovercraft



Reference path 2: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter

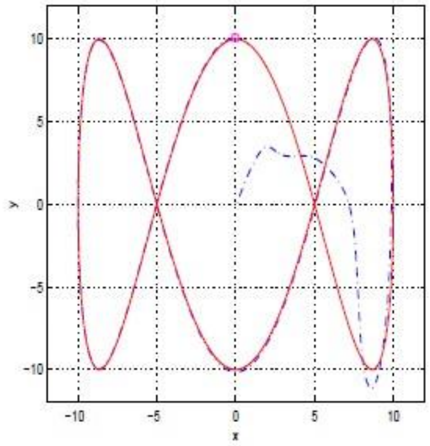


Reference path 3: Trajectory tracking for states x,y of the underactuated hovercraft

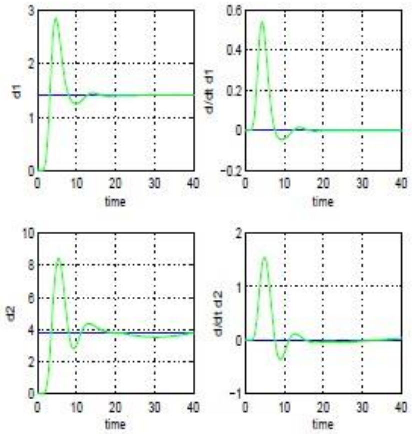


Reference path 3: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter

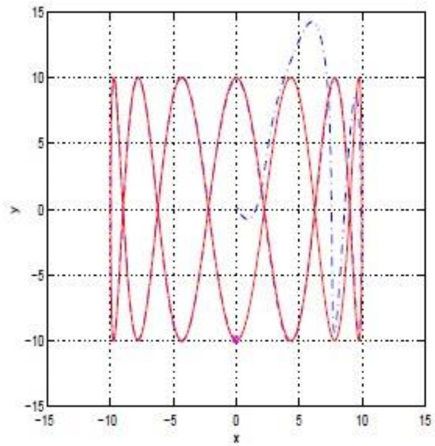
7. Simulation tests



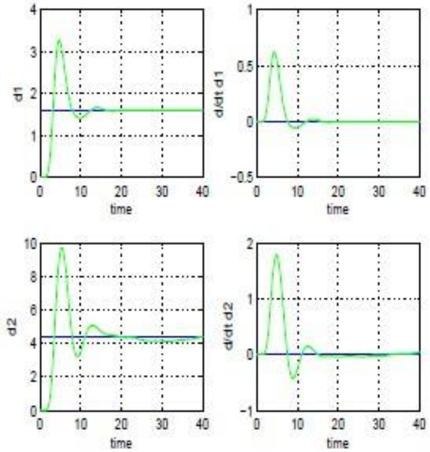
Reference path 4: Trajectory tracking for states x,y of the underactuated hovercraft



Reference path 4: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter



Reference path 5: Trajectory tracking for states x,y of the underactuated hovercraft



Reference path 5: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter

8. Conclusions

- A **nonlinear control method** has been developed for the **underactuated model** of an **unmanned surface vessel**, based on **differential flatness theory** and on a new nonlinear filtering method under the name **Derivative-free nonlinear Kalman Filter**. First, it was shown that the vessel's model is differentially flat.

- **Dynamic extension** has been used. The system has been augmented by considering as **additional state variables** the control inputs and their derivatives.

- By applying **dynamic extension** and **differential flatness properties**, the vessel's model has been transformed into a **linear form**. Moreover, using the linearized model of the vessel, a **state feedback controller** has been designed.

- Next, to estimate the **non-measurable state variables** of the vessel and to identify additive **disturbance terms** that affected he system, the **Derivative-free nonlinear Kalman Filter** was redesigned as a **disturbance observer.**

- This estimation algorithm consists of the standard Kalman Filter recursion applied on the linearized equivalent of the system and of an **inverse transformation** that is based on differential flatness theory which permits to compute estimates of the state variables of the initial nonlinear system.