Nonlinear control and state estimation for autonomous robotic vehicles

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1 . Outline

● The reliable functioning of autonomous robotic vehicles relies on the solution of the associated nonlinear control and state estimation problems

● The main approaches followed towards the solution of nonlinear control problem are as follows: (i) **control with global linearization** methods (ii) **control with approximate (asymptotic) linearization** methods (iii) **control with Lyapunov theory methods** (adaptive contro methods) when the dynamic or kinematic model of the robotic vehicle is unknown

● The main approaches followed towards the solution of the nonlinear state estimation problems are as follows: (i) state estimation with methods global linearization (ii) state estimation with methods of approximate (asymptotic) linearization

● Factors of major importance for the control loop of autonomous robotic vehicles are as follows (i) global stability conditions for the related nonlinear control scheme (ii) global stability conditions for the related nonlinear state estimation scheme (iii) global asymptotic stability for the joint control and state estimation scheme

2 . Nonlinear control and state estimation with global linearization

● To this end the differential flatness control theory is used

● The method can be applied to all nonlinear systems which are subject to an input-output linearization and actually such systems posses the property of differential flatness

● The state-space description for the dynamic or kinematic model of the robotic vehicle is transformed into a more compact form that is input-output linearized. This is achieved after defining the system’s flat outputs

● A system is differentially flat if the following two conditions hold: (i) all state variables and control inputs of the system can be expressed as differential functions of its flat outputs (ii) the flat outputs of the system and their time-derivatives are differentially independent, which means that they are not connected through a relying having the form of an ordinary differential equation

● With the applications of change of variables (diffeomorphisms) that rely on the differential flatness property (i), the state-space description of the robotic system is written into the linear canonical form. For the latter state-space description it is possible to solve both the control and the state estimation problem for the robotic vehicle.

3 . Nonlinear control and state estimation with approximate linearization

- To this end the theory of optimal H-infinity control and the theory of optimal H-infinity state estimation are used

- The nonlinear state-space description of the system undergoes approximate linearization around a temporary operating point which is updated at each iteration of the control and state estimation algorithm

- The linearization relies on first order Taylor series expansion around the temporary operating point and makes use of the computation of the associated Jacobian matrices

- The linearization error which is due to the truncation error of higher-order terms in the Taylor series expansion is considered to be a perturbation that is finally compensated by the robustness of the control algorithm

- For the linearized description of the state-space model an optimal H-infinity controller is designed. For the selection of the controller's feedback gains an algebraic Riccati equation has to be solved at each time step of the control algorithm

- Through Lyapunov stability analysis, the global stability properties of the control method are proven

- For the implementation of the optimal control method through the processing of measurements from a small number of sensors of the robotic vehicle, the H-infinity Kalman Filter is used as a robust state estimator

4 . Nonlinear control and state estimation with Lyapunov methods

● By initially proving the differential flatness properties for the robotic system and by defining its flat outputs a transformation of Its state-space description into an equivalent input-output linearized form is achieved.

● The unknown dynamics of the robotic vehicle is incorporated into the transformed control inputs of the system, which now appear in its equivalent input-output linearized state-space description

● The motion control problem for the robotic vehicle of unknown dynamics in now turned into a problem of indirect adaptive control. The computation of the control inputs of the system is performed simultaneously with the identification of the nonlinear functions which constitute its unknown dynamics.

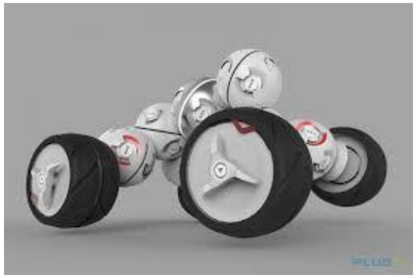
● The estimation of the unknown dynamics of the robotic vehicle is performed through the adaptation of neurofuzzy approximators. The definition of the learning parameters takes place through gradient algorithms of proven convergence, as demonstrated by Lyapunov stability analysis

● The Lyapunov stability method is the tool for selecting both the gains of the stabilizing feedback controller and the learning rate of the estimator of the unknown system's dynamics

● Equivalently through Lyapunov stability analysis the feedback gains of the state estimators of the robotic system are chosen. Such observers are included in the control loop so as to enable feedback control through the processing of a small number of sensor measurements

5 . Example 1: Nonlinear control and state estimation with global linearization

5.1.Overview



• **Controller design for autonomous 4-wheeled ground vehicles** is performed with **differential flatness theory**.

• Using a **3-DOF nonlinear model** of the vehicle's dynamics and through the application of differential flatness theory an equivalent model in **linear canonical** (Brunovksy) form is obtained.

• For the latter model a **state feedback controller** is developed that enables accurate tracking of velocity setpoints.

• Moreover, it is shown that with the use of **Kalman Filtering** it is possible to dynamically **estimate the effects of unknown disturbance forces exerted** on the vehicle.

• The processing of velocity measurements (provided by a small number of on-board sensors) through a **Kalman Filter** which has been redesigned in the form of a **disturbance observer** results in accurate **identification of external disturbances** affecting the vehicle's dynamic model.

• By including in the vehicle's controller an **additional term that compensates for the estimated disturbance forces**, the vehicle's motion characteristics remain unchanged.

• Numerical simulation confirms the **efficiency of** both the proposed **controller** and of the **disturbances estimator.**

5 . Example 1: Nonlinear control and state estimation with global linearization

5.2. Dynamic analysis of the four-wheel robotic vehicle

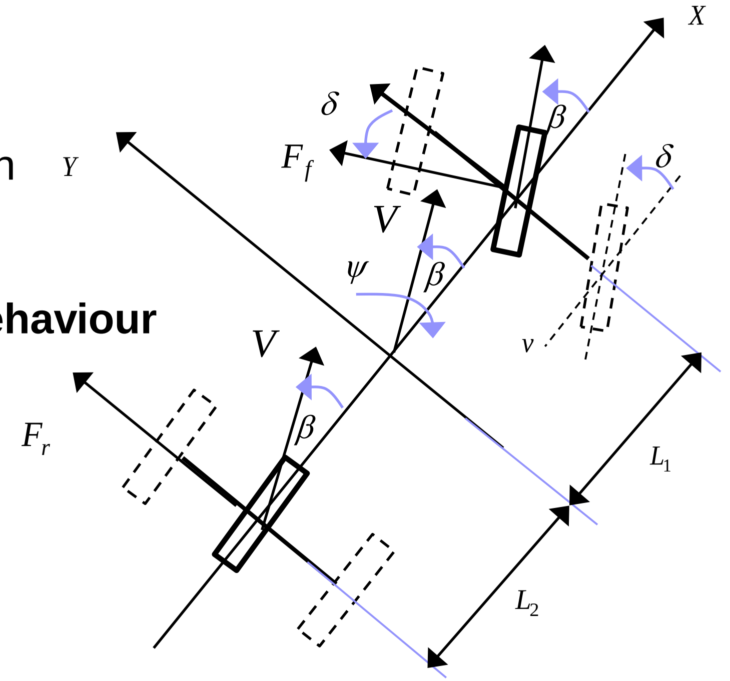
- The dynamic model of the vehicle **associates its acceleration** to the **forces and torques applied on it**, e.g. the force of the engine, friction and lateral forces on the tires, etc.

- The development of **elaborated dynamic models** of the vehicle can be particularly useful for the design of active safety systems. This can help in:

1) Lane keeping and avoidance of road departing when maneuvers are too demanding

2) **Control of both the lateral** and the **longitudinal behaviour of the vehicle.**

- A dynamic model of a 4-wheel vehicle can be:



$$\begin{bmatrix} -\sin\beta & \cos\beta & 0 \\ \cos\beta & \sin\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} mV(\dot{\beta}+\dot{\psi}) \\ m\dot{V} \\ I\ddot{\psi} \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ T_z \end{bmatrix}$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.2. Dynamic analysis of the four-wheel robotic vehicle

β : angle between velocity and the OX axis

V : velocity vector of the vehile

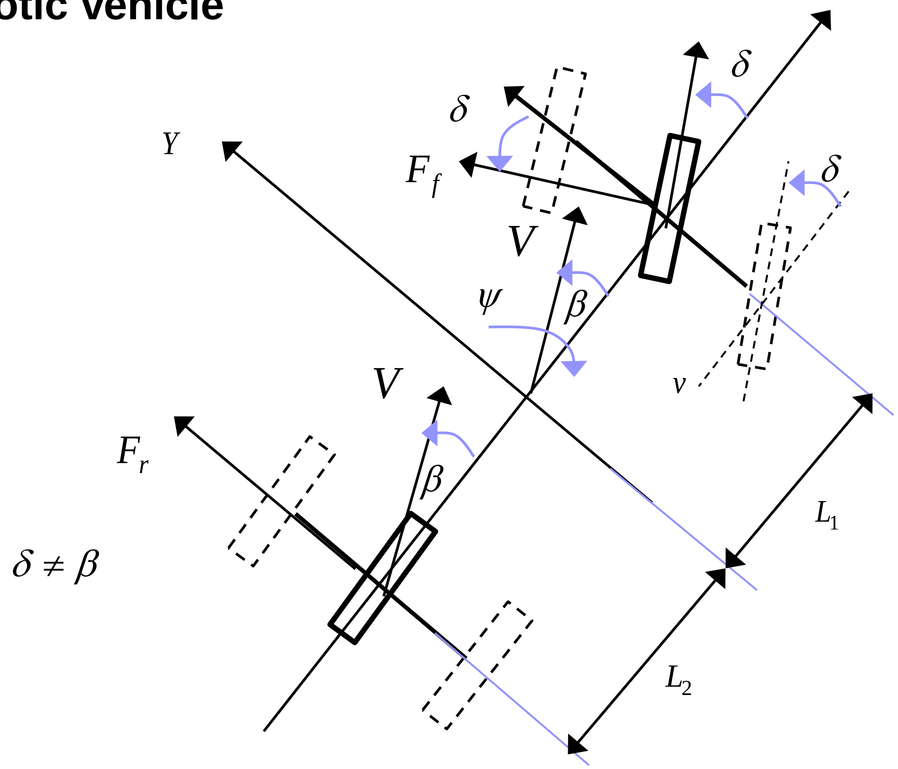
ψ : yaw (rotation round z axis)

f_x : aggregate force along x axis

f_y : aggregate force along y axis

T_z : torque round z axis

δ : steering angle of front wheels



- **longitudinal motion**

$$-mV(\dot{\beta}+\dot{\psi})\sin(\beta)+m\dot{V}\cos(\beta)=f_x$$

- **lateral motion**

$$mV(\dot{\beta}+\dot{\psi})\cos(\beta)+m\dot{V}\sin(\beta)=f_y$$

- **yaw turn**

$$I\ddot{\psi}=T_z$$

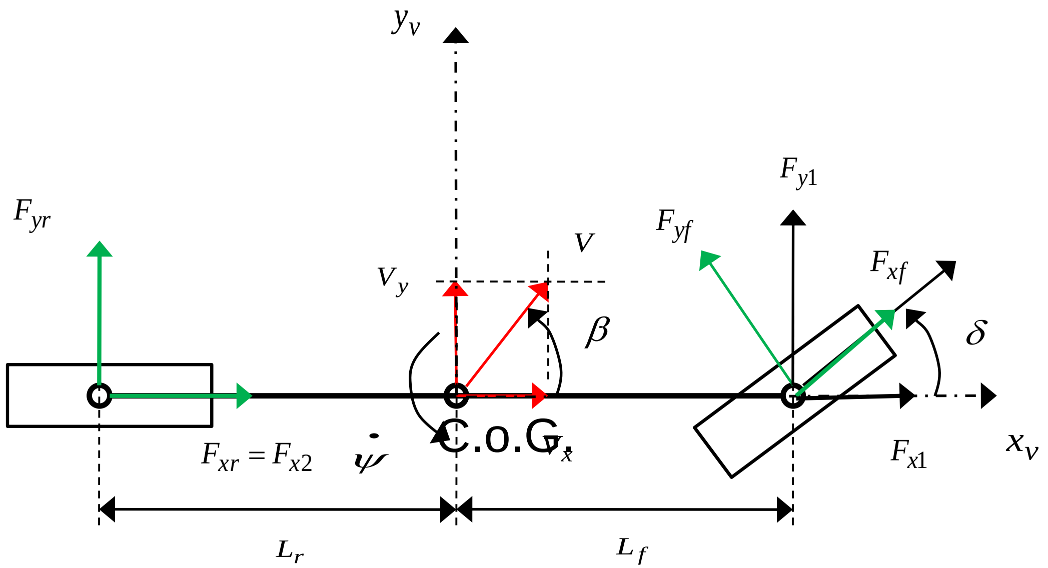
Forces on tires are transformed into forces and torques along the vehicle's axes:

$$\begin{bmatrix} f_x \\ f_y \\ T_z \end{bmatrix} = \begin{bmatrix} -\sin(\delta) & 0 \\ \cos(\delta) & 1 \\ L_1\cos(\delta) & -L_2 \end{bmatrix} \begin{bmatrix} F_f \\ F_r \end{bmatrix}$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.3. Dynamic analysis of the vehicle with longitudinal and transversal forces

The previous dynamical model of the vehicle is re-examined considering $\dot{\beta} = 0$



The **vehicle's dynamics** is formulated as:

$$ma_x = m(\dot{V}_x - \dot{\psi}\dot{V}_y) = F_{x_1} + F_{x_2}$$

$$ma_y = m(\dot{V}_y + \dot{\psi}\dot{V}_x) = F_{y_1} + F_{y_2}$$

$$I_z \ddot{\psi} = T_{z_1} + T_{z_2}$$

F_{x_i}, $i = 1,2$ forces applied on the longitudinal axis of the vehicle

F_{y_i}, $i = 1,2$ forces applied on the transversal axis of the vehicle

5 . Example 1: Nonlinear control and state estimation with global linearization

5.3. Dynamic analysis of the vehicle with longitudinal and transversal forces

The forces and torques which are exerted on the vehicle are defined as follows:

Forces along the vehicle's longitudinal axis:

$$F_{x_1} = F_{x_f}\cos(\delta) - F_{y_f}\sin(\delta)$$
$$F_{x_2} = F_{x_r}$$

Forces along the vehicle's transversal axis:

$$F_{y_1} = F_{x_f}\sin(\delta) + F_{y_f}\cos(\delta)$$
$$F_{y_2} = F_{y_r}$$

Torque's along the vehicle's z-axis:

$$T_{z_1} = L_f(F_{y_f}\cos(\delta) + F_{x_f}\sin(\delta))$$
$$T_{z_2} = -L_r F_{y_r}$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.3. Dynamic analysis of the vehicle with longitudinal and transversal forces

About the longitudinal and the lateral forces on the front and rear wheel one has:

1. Longitudinal force on the front wheel:

$$F_{x_f} = (\frac{1}{R})(I_r \dot{\omega}_f + T_m - T_{b_r})$$

T_m motor's torque transferred to the front wheels

T_{b_r} braking torque

2. Longitudinal force on the rear wheel:

$$F_{x_r} = -(\frac{1}{R})(T_{b_r} - I_r \dot{\omega}_r)$$

3. Lateral force on the front wheel:

$$F_{y_f} = C_f(\delta - \beta_f) = C_f(\delta - \frac{V_y + \dot{\psi} L_f}{V_x})$$

where the wheel's sideslip angle is $\beta_f = \frac{V_y + \dot{\psi} L_f}{V_x}$

and the wheel's turn angle is δ

4.Lateral force on the rear wheel:

$$F_{y_r} = -C_r \frac{V_y - \dot{\psi} L_f}{V_x}$$

taking that the sideslip angle is $\beta_r = \frac{V_y - \dot{\psi} L_f}{V_x}$

and $\delta = 0$

C_f, C_r: **cornering stiffness coefficients** for front and rear wheels

5 . Example 1: Nonlinear control and state estimation with global linearization

5.3. Dynamic analysis of the vehicle with longitudinal and transversal forces

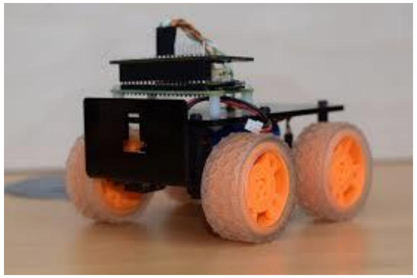
The **vehicle's dynamics** is described by the following set of differential equations:

$$m\dot{V}_x = m\dot{\psi}V_y - \frac{I_r}{R}(\dot{\omega}_r + \dot{\omega}_f + \frac{1}{R}(T_m - T_{b_f} - T_{b_r}) + C_f(\frac{V_y + \dot{\psi}L_f}{V_x})\delta - C_f\delta^2$$

$$m\dot{V}_y = -m\dot{\psi}V_x - C_f(\frac{V_y + \dot{\psi}L_f}{V_x}) - C_r(\frac{V_y - \dot{\psi}L_f}{V_x}) + \frac{1}{R}(T_m - T_{b_f})\delta + (C_f - \frac{I_r}{R}\dot{\omega}_f)\delta$$

$$I_z\ddot{\psi} = -L_fC_f(\frac{V_y + \dot{\psi}L_f}{V_x}) + L_rC_r(\frac{V_y - \dot{\psi}L_f}{V}) + \frac{L_f}{R}(T_m - T_{b_f})\delta + L_f(T_m - \frac{I_r}{R})\delta$$

The **two control inputs** to the vehicle's dynamic model are:



$$u_1 = T_\omega = T_m - (T_{b_f} + T_{b_r})$$

$$u_2 = \delta$$

A first form of the **state-space equation** of vehicle's dynamic model is:

$$\dot{x} = f(x,t) + g(x,t)u + g_1u_1u_2 + g_2u_2^2$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.3. Dynamic analysis of the vehicle with longitudinal and transversal forces

The **nonlinear state-space equation** of the vehicle comprises the following elements:

$$\dot{x} = f(x,t) + g(x,t)u + g_1 u_1 u_2 + g_2 u_2^2$$

where

$$f(x,t) = \begin{pmatrix} \frac{I_r}{mR}(\dot{\omega}_r + \dot{\omega}_f) \\ \dot{\psi} V_x + \frac{1}{m}\left(-C_f \frac{(V_y + L_f \dot{\psi})}{V_x} - C_r \frac{(V_y - L_f \dot{\psi})}{V_x}\right) \\ \frac{1}{I_z}\left(-L_f C_f \frac{(V_y + L_f \dot{\psi})}{V_x} + L_r C_r \frac{(V_y - L_f \dot{\psi})}{V_x}\right) \end{pmatrix} \qquad g(x,t) = \begin{pmatrix} \frac{1}{mR} & \frac{C_f}{m}\left(\frac{V_y + L_f \dot{\psi}}{V_x}\right) \\ 0 & \frac{C_f R - I_r \dot{\omega}_f}{mR} \\ 0 & \frac{L_f C_f R - L_f I_r \dot{\omega}_f}{I_z R} \end{pmatrix}$$

$$g_1 = \begin{pmatrix} 0 \\ \frac{1}{mR} \\ \frac{L_f}{I_z R} \end{pmatrix} \qquad g_2 = \begin{pmatrix} \frac{-C_f}{m} \\ 0 \\ 0 \end{pmatrix} \qquad x = \begin{pmatrix} V_x \\ V_y \\ \dot{\psi} \end{pmatrix} \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

By omitting terms $u_1 u_2$ and u_2^2 the model is simplified into

$$\dot{x} = f(x,t) + g(x,t)u$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.4. Flatness-based control for the 3-DOF model of the robotic vehicle

5.4.1. Differential flatness theory for dynamical systems

- A dynamical system can be written in the ODE form $S_i(w,\dot{w},\ddot{w},\dots,w^{(i)}),\ \ i=1,2,\dots,q$

  where $w^{(i)}$ stands for the i-th derivative of either a state vector element or of a control input

- The system is said to be **differentially flat** with respect to the **flat output**

$$y_i=\phi(w,\dot{w},\ddot{w},\dots,w^{(a)}),\ i=1,\dots,m$$

where $\quad y=(y_1,y_2,\dots,y_m)$

if the following two conditions are satisfied

(i) There does not exist any differential relation of the form

$$R(y,\dot{y},\ddot{y},\dots,y^{(\beta)})=0$$

which means that **the flat output and its derivatives are not coupled**

(ii) All system variables are **functions of the flat output and its derivatives**

$$w^{(i)}=\psi(y,\dot{y},\ddot{y},\dots,y^{(\gamma_i)})$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.4. Flatness-based control for the 3-DOF model of the robotic vehicle

5.4.1. Differential flatness theory for dynamical systems

- Thus, differential flatness means that that **all system dynamics**

  can be expressed as a **function of a flat output and its derivatives**, i.e.

  If the dynamic system is initially written as

$$\dot{x} = f(x,u),\ \ x \in R^n, u \in R^m$$

where x is the state vector, u is the control input, and y is the flat output

then one can find functions ϕ,ψ such that

$$x = \phi(y, \dot{y}, ..., y^{(r-1)}) \qquad \phi : (R^m)^r \to R^n$$

$$u = \psi(y, \dot{y}, ..., y^{(r-1)}, y^{(r)}) \qquad \psi : (R^m)^{r+1} \to R^m$$

- For linear systems the property of **differential flatness** coincides with that of **controllability**

- The concept of differential flatness can be also extended to **distributed parameter systems**

5 . Example 1: Nonlinear control and state estimation with global linearization

5.4. Flatness-based control for the 3-DOF model of the robotic vehicle

5.4.2. Classes of differentially-flat systems

1. **Affine in the input systems** $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$

the above state equation can also describe MIMO dynamical systems.

The generic class of systems $\dot{x} = f(x,u)$ can be also transformed into an affine-in-the-input form

2. **Driftless systems** $\dot{x} = \sum_{i=1}^{m} f_i(x) u_i$

For driftless systems with two inputs, i.e. $\dot{x} = f_1(x)u_1 + f_2(x)u_2$

the flatness property holds if and only if.

rank of matrix $E_{k+1} := \{E_k, [E_k, E_k]\}, k \geq 0$ with
$E_0 := \{f_1, f_2\}$ is equal to $k+2 \quad k = 0,\ldots,n-2$

For driftless systems with n-2 inputs, i.e. $\dot{x} = \sum_{i=1}^{n-2} f_i(x) u_i \quad x \in R^n$

the flatness property holds, if controllability also holds. Furthermore, the system is 0-flat, i.e. the flat output is a function of only the state vector elements x_i , if *n is even*

5 . Example 1: Nonlinear control and state estimation with global linearization

5.4. Flatness-based control for the 3-DOF model of the robotic vehicle

5.4.3. Transformation of nonlinear systems into a canonical form

● To define **necessary and sufficient conditions** for the existence of a diffeomorphism that transforms the initial nonlinear system into the canonical (Brunovsky) form the following definitions are used:

(i) **Lie derivative**

$L_f(x)$ stands for the Lie derivative $L_f h(x) = (\nabla h) f$

and the repeated Lie derivatives are recursively defined as

$L_f^0 h = h$ for $i = 0$, $L_f^i h = L_f L_f^{i-1} h = \nabla L_f^{i-1} h f$ for $i = 1,2,...$

(ii) **Lie bracket**

$ad_f^i g$ stands for a Lie bracket

which is defined recursively as $ad_f^i g = [f, ad_f^{i-1} g]$

with $ad_f^0 g = g$ and $ad_f g = [f, g] = \nabla_g f - \nabla_f g$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.4. Flatness-based control for the 3-DOF model of the robotic vehicle

5.4.3. Transformation of nonlinear systems into a canonical form

● Necessary and sufficient conditions for transforming MIMO systems into the **canonical form** after applying differential flatness theory (S. Bououden, D. Boutat, G. Zheng, J.P. Barbot and F. Kratz, 2011)

A MIMO system of the following form is considered $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$ A

If the system of A can be linearized by a diffeomorphism $z = \phi(x)$ and a static state feedback $u = a(x) + \beta(x)v$ into the following form

$$\dot{z}_{i,j} = z_{i+1,j} \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq v_j - 1$$

$$\dot{z}_{v_{i,j}} = v_j$$

B

with $\sum_{j=1}^{m} v_j = n,$ then $y_j = z_{1,j}$ for $1 \leq j \leq m$

are the 0-flat outputs which can be written as functions of only the elements of the state vector x..

5 . Example 1: Nonlinear control and state estimation with global linearization

5.4. Flatness-based control for the 3-DOF model of the robotic vehicle

5.4.3. Transformation of nonlinear systems into a canonical form

To define conditions for transforming the system of A into the canonical form described in B the following theorem holds

Theorem:

For the nonlinear systems described by $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$ the following variables are defined

$$(i) G_0 = span[g_1, ..., g_m]$$

$$(ii) G_1 = span[g_1, ..., g_m, ad_f g_1, ..., ad_f g_m]$$

$$...$$

$$(iii) G_k = span[ad_f^j g_i \quad \text{for} \quad 0 \le j \le k, 1 \le i \le m]$$

Then the linearization problem for the above class of systems can be solved if and only if

(1) the dimension of $G_i, i = 1, ..., k$ is constant for $x \in X \subset R^n$ and for $1 \le i \le n-1$

(2) the dimension of G_{n-1} is of order n

(3) the distribution G_k is involutive for each $1 \le k \le n-2$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.4. Flatness-based control for the 3-DOF model of the robotic vehicle

5.4.3. Transformation of nonlinear systems into a canonical form

It is assumed now that after expressing the system state variables and control inputs as functions of the flat output and of the associated derivatives, the system can be transformed in the **Brunovsky canonical form**:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\cdots \\
\dot{x}_{r_1-1} &= x_{r_1} \\
\dot{x}_{r_1} &= f_1(x) + \sum_{j=1}^{p} g_{1_j}(x) u_j + d_1 \\
\\
\dot{x}_{r_1+1} &= x_{r_1+2} \\
\dot{x}_{r_1+2} &= x_{r_1+3} \\
&\cdots \\
\dot{x}_{p-1} &= x_p \\
\dot{x}_p &= f_p(x) + \sum_{j=1}^{p} g_{p_j}(x) u_j + d_p \\
\\
y_1 &= x_1 \\
y_2 &= x_2 \\
&\cdots \\
y_p &= x_{n-r_p+1}
\end{aligned}
$$

Having written the initial nonlinear system into the canonical (Brunovsky) form it holds

$$y_i^{(r_i)} = f_i(x) + \sum_{j=1}^{p} g_{ij}(x) u_j + d_j$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.4. Flatness-based control for the 3-DOF model of the robotic vehicle

5.4.3. Transformation of nonlinear systems into a canonical form

Next the following vectors and matrices can be defined

$$f(x)=[f_1(x), \quad ..., \quad f_n(x)]^T$$

$$g(x)=[g_1(x), \quad ..., \quad g_n(x)]^T$$

with $g_i(x)=[g_{1i}(x), \quad ..., \quad g_{pi}(x)]^T$

$$A=diag[A_1,...,A_p], \ B=diag[B_1,...,B_p]$$

$$C^T=diag[C_1,...,C_p], \ d=[d_1,...,d_p]^T$$

where matrix A has the **MIMO canonical form**, i.e. with elements

$$A_i=\begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{r_i \times r_i}$$

$$B_i^T=[0 \quad 0 \quad ... \quad 0 \quad 1]_{1\times r_i} \qquad C_i=[1 \quad 0 \quad ... \quad 0 \quad 0]_{1\times r_i}$$

Thus, the initial nonlinear system can be written in the state-space form

$$\dot{x}=Ax+B[f(x)+g(x)u+\tilde{d}]$$
$$y=Cx$$

or equivalently in the state space form

$$\dot{x}=Ax+Bv+B\tilde{d}$$
$$y=Cx$$

where $v=f(x)+g(x)u$

For the case of the multi-DOF MIMO robotic model it is assumed that the functions $f(x)$ and $g(x)$ are known and due to missing sensory information some of its state vector elements are not measurable

5 . Example 1: Nonlinear control and state estimation with global linearization

5.5. Design of a flatness-based controller for the 3-DOF model of the vehicle

To show the **differential flatness of the vehicle's model** the following flat outputs are defined:

$$y_1 = V_x$$

$$y_2 = L_f m V_y - I_z \dot{\psi}$$

All elements of the system's state vector $x = \begin{bmatrix} V_x & V_y & \dot{\psi} \end{bmatrix}$ can be written as functions of the **flat output** and of its derivatives

$$V_x = y_1$$

$$V_y = \frac{y_2}{L_f m} - \left(\frac{I_z}{L_f m}\right)\left(\frac{L_f m y_1 \dot{y}_2 + C_r(L_f + L_r) y_2}{C_r(L_f + L_r)(I_z - L_f L_r m) + (L_f m y_1)^2}\right)$$

$$\dot{\psi} = \left(\frac{L_f m y_1 \dot{y}_2 + C_r(L_f + L_r) y_2}{C_r(L_f + L_r)(I_z - L_f L_r m) + (L_f m y_1)^2}\right)$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.5. Design of a flatness-based controller for the 3-DOF model of the vehicle

Expressing the system's state variables as functions of the flat output and their derivatives one obtains the **following state-space description**

$$\begin{bmatrix} \dot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \Delta(y_1, y_2, \dot{y}_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \Phi(y_1, y_2, \dot{y}_2)$$

where

$$\Delta(y_1, y_2, \dot{y}_2) = \begin{bmatrix} \Delta_{11}(y_1, y_2, \dot{y}_2) & \Delta_{12}(y_1, y_2, \dot{y}_2) \\ \Delta_{21}(y_1, y_2, \dot{y}_2) & \Delta_{22}(y_1, y_2, \dot{y}_2) \end{bmatrix}$$

with

$$\Delta_{11}(y_1, y_2, \dot{y}_2) = \frac{1}{mR} \qquad \Delta_{12}(y_1, y_2, \dot{y}_2) = \frac{C_f}{m}\left(\frac{V_y + L_f \dot{\psi}}{y_1}\right)$$

$$\Delta_{21}(y_1, y_2, \dot{y}_2) = \frac{C_r(L_f + L_r)(V_y - L_r \dot{\psi}) - L_f m \dot{\psi} y_1^2}{mRy_1^2}$$

$$\Delta_{22}(y_1, y_2, \dot{y}_2) = \left(-L_f m y_1 + \frac{L_r C_r (L_f + L_r)}{y_1}\right) \frac{(L_f C_f R - L_f I_r \dot{\omega}_r)}{I_z R} +$$

$$+ \frac{(C_r(L_f + L_r))(V_y - L_r \dot{\psi}) - L_f m \dot{\psi} y_1^2}{y_1^2} \frac{C_f (V_y + L_r \dot{\psi})}{m y_1} - \frac{C_r (L_f + L_r)}{y_1} \frac{RC_f - I_r \dot{\omega}_f}{mR}$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.5. Design of a flatness-based controller for the 3-DOF model of the vehicle

About matrix $\Phi(y_1, y_2, \dot{y}_2)$ it holds

$$\Phi(y_1, y_2, \dot{y}_2) = \begin{bmatrix} \Phi_1(y_1, y_2, \dot{y}_2) \\ \Phi_2(y_1, y_2, \dot{y}_2) \end{bmatrix}$$

with

$$\Phi_1(y_1, y_2, \dot{y}_2) = \dot{\psi} V_y - \frac{I_r}{mR}(\dot{\omega}_r + \dot{\omega}_f)$$

$$\Phi_2(y_1, y_2, \dot{y}_2) = -L_f m y_1 f_3(x,t) - \frac{C_r(L_f + L_r)}{y_1} f_2(x,t) +$$

$$+ \frac{C_f(L_f + L_r)(V_y - L_r \dot{\psi}) - L_f m \dot{\psi} y_1^2}{y_1^2} f_1(x,t) - \frac{L_r C_r (L_f + L_r)}{y_1} f_3(x,t)$$

According to the above the system’s control input can be also expressed as a **function of the flat output and its derivatives**

5 . Example 1: Nonlinear control and state estimation with global linearization

5.5. Design of a flatness-based controller for the 3-DOF model of the vehicle

Thus one has

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \Delta(y_1, y_2, \dot{y}_2)^{-1}\left(\begin{bmatrix} \dot{y}_1 \\ \ddot{y}_2 \end{bmatrix} - \Phi(y_1, y_2, \dot{y}_2)\right)$$

while conditions for the **non-singularity** of matrix $\Delta(y_1, y_2, \dot{y}_2)$ are also proven to hold

Indeed, the determinant

$$\det(\Delta(y_1, y_2, \dot{y}_2)) = \frac{(I_r \dot{\omega}_f - C_f R)(L_f^2 y_1^2 m^2 - C_r(L_f + L_r)L_r L_f m + C_r I_z L_r)}{I_z R^2 y_1 m^2}$$

is non-zero, because it holds

(i) $(I_r \dot{\omega}_f - C_f R) \neq 0$ since for the wheels rotational acceleration one has $\dot{\omega}_f < \frac{C_f R}{I_r}$

(ii) $(L_f^2 y_1^2 m^2 - C_r(L_f + L_r)L_r L_f m + C_r I_z L_r) \neq 0$ when $I_z > L_f m$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.5. Design of a flatness-based controller for the 3-DOF model of the vehicle

The **differentially flat model** can be written in canonical form after defining the control input

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \Delta(y_1, y_2, \dot{y}_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \Phi(y_1, y_2, \dot{y}_2)$$

Then one obtains the description of the vehicle's model in the **MIMO canonical form**

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The control law which assures convergence to the desirable velocity setpoints is

$$v_1 = \dot{y}_1^{ref} - k_{p_1}(y_1 - y_1^{ref})$$

$$v_2 = \ddot{y}_2^{ref} - k_{d_2}(\dot{y}_2 - \dot{y}_2^{ref}) - k_{p_2}(y_2 - y_2^{ref})$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.5. Design of a flatness-based controller for the 3-DOF model of the vehicle

Defining the error variables $e_1 = y_1 - y_1^{ref}$ and $e_2 = y_2 - y_2^{ref}$

the **tracking error dynamics** for suitable selection of feedback gains becomes

$$\dot{e}_1 + k_{p_1} e_1 = 0 \Rightarrow \lim_{t \to \infty} e_1(t) = 0$$

$$\ddot{e}_2 + k_{d_2} \dot{e}_2 + k_{p_2} e_2 = 0 \Rightarrow \lim_{t \to \infty} e_2(t) = 0$$

The **control input** that is finally applied to the vehicle is given by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \Delta^{-1}(y_1, y_2, \dot{y}_2)\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \Phi(y_1, y_2, \dot{y}_2)\right)$$

or equivalently

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \Delta^{-1}(y_1, y_2, \dot{y}_2)\left(\begin{bmatrix} \dot{y}_1^{ref} - k_{p_1}(y_1 - y_1^{ref}) \\ \ddot{y}_2^{ref} - k_{d_2}(\dot{y}_2 - \dot{y}_2^{ref}) - k_{p_2}(y_2 - y_2^{ref}) \end{bmatrix} - \Phi(y_1, y_2, \dot{y}_2)\right)$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.6. State estimation with the use of differential flatness theory

The continuous-time Kalman Filter

- For mechatronic systems with linear dynamics the **Kalman Filter is the optimal state estimator** since it can provide estimates of the state vector elements of maximum accuracy (minimum variance) through the processing of measurements from a small number of sensors.

- For **the continuous-time dynamical system**

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$y(t) = Cx(t) + v(t)$$

where $E(w(t)w(t+\tau)) = Q(t)$ and $E(v(t)v(t+\tau)) = Q(t)$

the **Kalman Filter** is a **state observer** which is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K[y - C\hat{x}(t)]$$
$$K = PC^T R^{-1}$$
$$\dot{P} = AP + PA^T + Q - PC^T R^{-1} CP \qquad \text{(Riccati Equation)}$$

where $\hat{x}$ is the state vector estimation

P is the estimation error covariance matrix

5 . Example 1: Nonlinear control and state estimation with global linearization

5.6. State estimation with the use of differential flatness theory

The discrete-time Kalman Filter

- The **discrete-time Kalman Filter** is an **optimal state estimator** for linear dynamical systems of the form:

$$x(k+1) = \Phi(k)x(k) + L(k)u(k) + w(k)$$
$$y(k) = Cx(k) + v(k)$$

$w(k)$: process noise $v(k)$: measurement noise

- The **process and measurement noises** are **uncorrelated Gaussian zero-mean** signals and their covariance matrices are:

$$Q = E[w(i)w^T(j)] \qquad R = E[v(i)v^T(j)]$$

- The initial values for the state vector estimation and for the covariance matrix of the estimation error are taken to be:

$\hat{x}(0) =$ a guess of $E[x(0)]$

$\hat{P}(0) =$ a guess of e.g $\hat{P}(0) = \lambda I$ with $\lambda > 0$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.6. State estimation with the use of differential flatness theory

The discrete-time Kalman Filter

● The Kalman filter can be decomposed into two parts:

i) **measurement update**: the set of measurements $Y=\{y(1),...,y(k-1),y(k)\}$ available

The estimation of $x(k)$ is $\hat{x}(k)$

$$K(k)=P^{-}(k)C^{T}[CP^{-}(k)C^{T}+R]^{-1}$$

$$\hat{x}(k)=\hat{x}^{-}(k)+K(k)[y(k)-C\hat{x}^{-}(k)]$$

$$P(k)=P^{-}(k)-K(k)CP^{-}(k)$$

ii) **Time update**: while measurement $y(k+1)$ has not been obtained yet

The a-priori estimation of $x(k)$ is $\hat{x}^{-}(k)$

$$P^{-}(k+1)=\Phi(k)P(k)\Phi^{T}(k)+Q(k)$$

$$\hat{x}^{-}(k+1)=\Phi\hat{x}(k)+Lu(k)$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.7. Disturbance estimation on the vehicle with the use of the Kalman Filter

5.7.1. State estimation with the Derivative-Free nonlinear Kalman Filter

It was shown that the vehicle's model can be written in the **MIMO canonical form**

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{A}$$

Thus one has a MIMO linear model of the form

$$\dot{y}_f = A_f y_f + B_f v$$
$$z_f = C_f y_f$$

where $y_f = [y_1, y_2, \dot{y}_2]^T$

and matrices A_f, B_f, C_f are defined as

$$A_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad C_f^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{B}$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.7. Disturbance estimation on the vehicle with the use of the Kalman Filter

5.7.1. State estimation with the Derivative-Free nonlinear Kalman Filter

The **measurable variables** are assumed to be $y_1 = V_x$ and $y_2 = L_f m V_y - I_z \dot{\psi}$

which in turn are associated with the measurement of linear and angular velocities $V_x, V_y, \dot{\psi}$

For the model of A and B and after discretization of matrices A_f, B_f, C_f

which results into A_f^d, B_f^d, C_f^d

one can estimate the state vector of the nonlinear vehicle's model by applying the **Kalman Filter recursion** to its equivalent linear canonical form.

This is the **Derivative-free nonlinear Kalman Filtering**

KF measurement update

$$K(k) = P^-(k) C_f^{d^T} [C_f^d P^-(k) C_f^{d^T} + R]^{-1}$$

$$\hat{x}(k) = \hat{x}^-(k) + K(k)[y(k) - C\hat{x}^-(k)]$$

$$P(k) = P^-(k) - K(k) C_f^d P^-(k)$$

KF time update

$$P^-(k+1) = A_f^d(k) P(k) A_f^{d^T}(k) + Q(k)$$

$$\hat{x}^-(k+1) = A_f^d \hat{x}(k) + B_f^d u(k)$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.7. Disturbance estimation on the vehicle with the use of the Kalman Filter

5.7.2. Modelling and estimation of disturbances in real-time

It is assumed that **disturbances forces** affect the vehicle along **its longitudinal and transversal axis** and that disturbance torques appear along its z-axis.

The disturbances dynamics are represented as

$$\tilde{d}_x = f_{d_x}(V_x, V_y, \dot{\psi})$$
$$\tilde{d}_y = f_{d_y}(V_x, V_y, \dot{\psi})$$
$$\tilde{d}_\psi = T_{d_\psi}(V_x, V_y, \dot{\psi})$$

The i-th order derivatives of the disturbances are denoted as

$$\tilde{d}_x^{(i)} = f_{d_x}^{(i)}(V_x, V_y, \dot{\psi})$$
$$\tilde{d}_y^{(i)} = f_{d_y}^{(i)}(V_x, V_y, \dot{\psi})$$
$$\tilde{d}_\psi^{(i)} = T_{d_\psi}^{(i)}(V_x, V_y, \dot{\psi})$$

Considering the effect of disturbance functions on the initial nonlinear state equation of the vehicle and the linear relation between state variables $[V_x, V_y]$ and the state variables of the flat system description one has the **appearance of the disturbance terms in the canonical form model**

5 . Example 1: Nonlinear control and state estimation with global linearization

5.7. Disturbance estimation on the vehicle with the use of the Kalman Filter

5.7.2. Modelling and estimation of disturbances in real-time

Canonical form representation of the vehicle's model **including disturbance terms**

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{pmatrix} \frac{1}{m}\tilde{d}_x \\ 0 \\ L_f \dot{\tilde{d}}_y - \dot{\tilde{d}}_\psi \end{pmatrix}$$

To obtain simultaneous estimation of the system's state vector of the disturbance terms the state vector of the system is extended to include also disturbances:

Next, the state vector of the model is extended to include as **additional state variables** the disturbances $\frac{1}{m}\tilde{d}_x$ and $L_f \dot{\tilde{d}}_y - \dot{\tilde{d}}_\psi$

The new state-space description comprises the state-space variables:

$$z_1 = y_1,\ z_2 = y_2,\ z_3 = \dot{y}_2,\ z_4 = \tilde{f}_a = \frac{1}{m}\tilde{d}_x$$

$$z_5 = \dot{\tilde{f}}_a,\ z_6 = \tilde{f}_b = L_f \dot{\tilde{d}}_y - \dot{\tilde{d}}_\psi,\ z_7 = \dot{\tilde{f}}_b$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.7. Disturbance estimation on the vehicle with the use of the Kalman Filter

5.7.2. Modelling and estimation of disturbances in real-time

With the **definition of the extended state vector** the state-space equation of the vehicle takes the form:

$$\dot{z} = \tilde{A} z + \tilde{B} \tilde{v}$$

where:

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \tilde{C}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where the measurable state variables are z_1 and z_2.

Since the dynamics of the disturbance terms $\tilde{f}_a$ and $\tilde{f}_b$ are taken to be unknown in the design of the associated **disturbances' estimator** one has the following dynamics

5 . Example 1: Nonlinear control and state estimation with global linearization

5.7. Disturbance estimation on the vehicle with the use of the Kalman Filter

5.7.2. Modelling and estimation of disturbances in real-time

Dynamics of the disturbances estimator:

$$\dot{z}_o = \tilde{A}_o z + \tilde{B}_o \tilde{v} + K(C_o z - C_o \hat{z})$$

where $K \in R^{7\times 2}$ is the state estimator's gain and

$$\tilde{A}_o = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{B}_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{C}_o^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Defining as $\tilde{A}_d, \tilde{B}_d, \tilde{C}_d$ the discrete-time equivalents of matrices $\tilde{A}_o, \tilde{B}_o, \tilde{C}_o$
a **Derivative-free nonlinear Kalman Filter** can be used for simultaneous estimation of the vehicle's state vector and of the unknown disturbances

5 . Example 1: Nonlinear control and state estimation with global linearization

5.7. Disturbance estimation on the vehicle with the use of the Kalman Filter

5.7.2. Modelling and estimation of disturbances in real-time

The **Derivative-free nonlinear Kalman Filter** for simultaneous state estimation and disturbances estimation is given by:

KF measurement update

$$K(k) = P^{-}(k)\tilde{C}_d^T[\tilde{C}_d P^{-}(k)\tilde{C}_d^T + R]^{-1}$$

$$\hat{z}(k) = \hat{z}^{-}(k) + K(k)[z(k) - \tilde{C}_d \hat{z}^{-}(k)]$$

$$P(k) = P^{-}(k) - K(k)\tilde{C}_d P^{-}(k)$$

KF time update

$$P^{-}(k+1) = \tilde{A}_d(k)P(k)\tilde{A}_d^T(k) + Q(k)$$

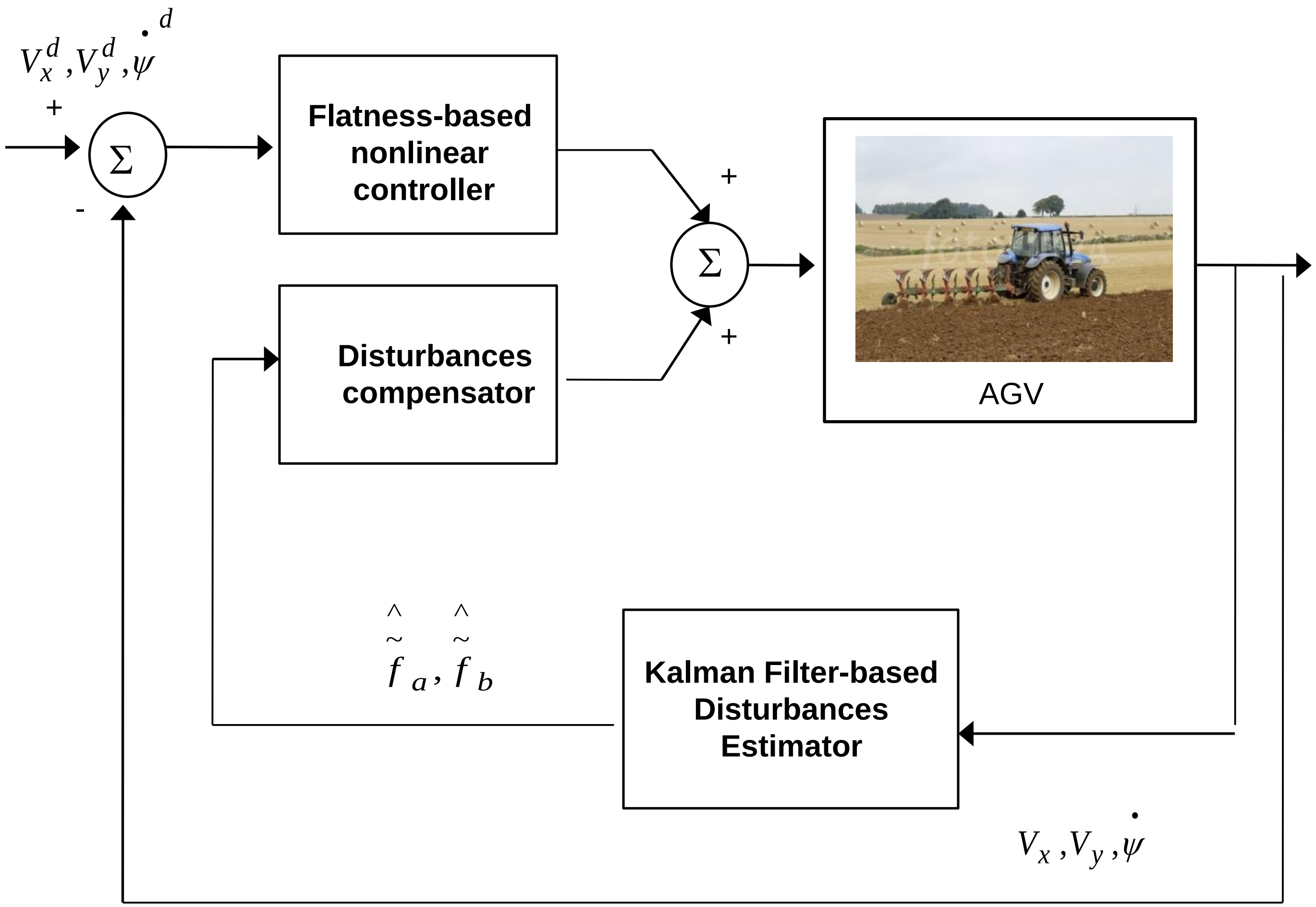
$$\hat{z}^{-}(k+1) = \tilde{A}_d \hat{z}(k) + \tilde{B}_d \tilde{v}(k)$$

To compensate for the effects of the disturbance forces the control input applied to the vehicle becomes

$$v = \begin{pmatrix} v_1 - \hat{\tilde{f}}_a \\ v_2 - \frac{d}{dt}\hat{\tilde{f}}_b \end{pmatrix} \qquad v = \begin{pmatrix} v_1 - \hat{z}_4 \\ v_2 - \hat{z}_6 \end{pmatrix}$$

5 . Example 1: Nonlinear control and state estimation with global linearization

5.8. Simulation results

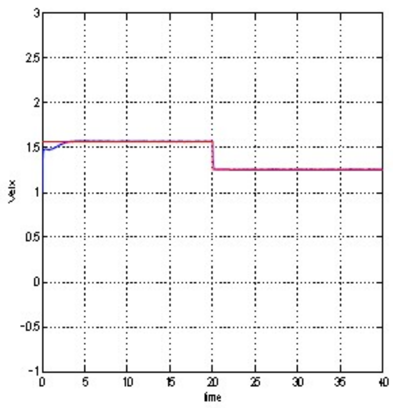


The control loop comprises (i) a flatness-based nonlinear controller
(ii) a Kalman Filter-based disturbances estimator
(iii) a disturbances compensator

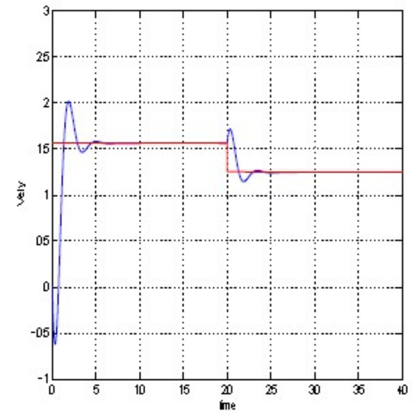
5 . Example 1: Nonlinear control and state estimation with global linearization

5.8. Simulation results

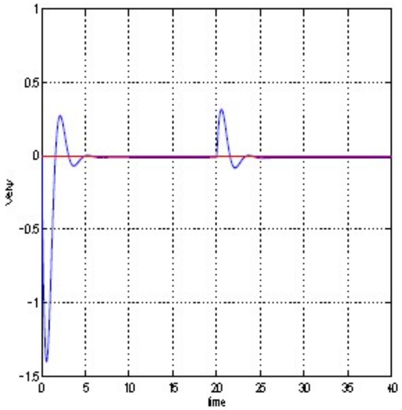
Disturbances profile 1:



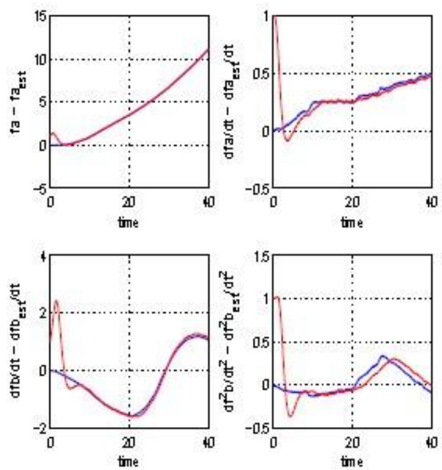
Control of x-axis velocity



Control of y-axis velocity



Control of rotational velocity

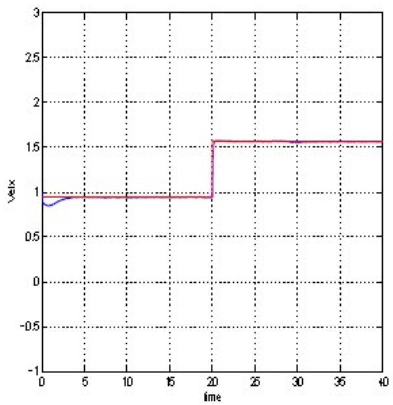


Disturbances estimation

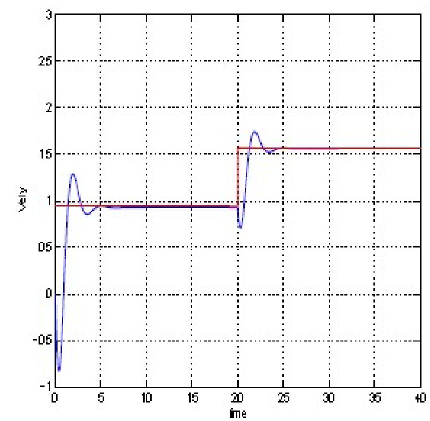
5 . Example 1: Nonlinear control and state estimation with global linearization

5.8. Simulation results

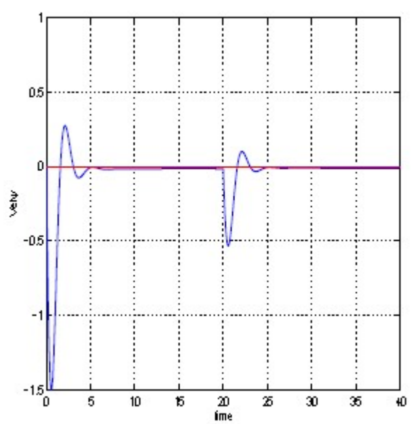
Disturbances profile 2:



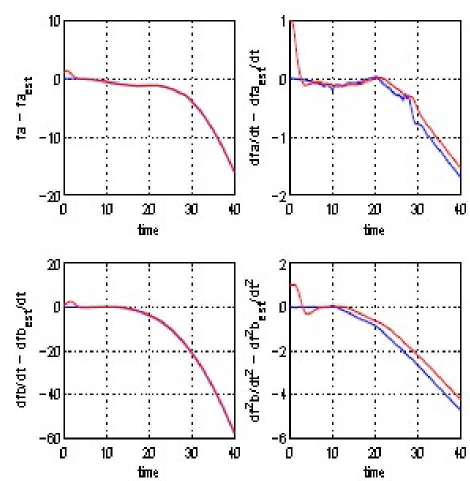
Control of x-axis velocity



Control of y-axis velocity



Control of rotational velocity

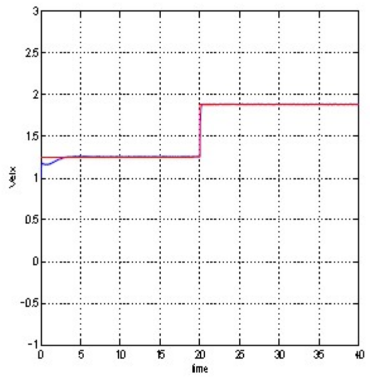


Disturbances estimation

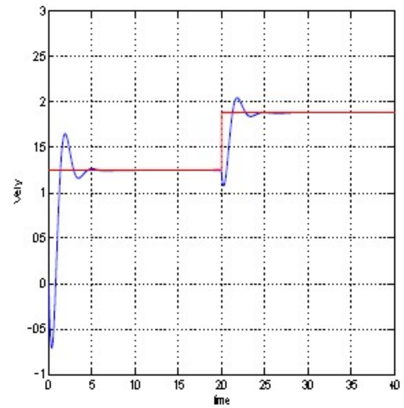
5 . Example 1: Nonlinear control and state estimation with global linearization

5.8. Simulation results

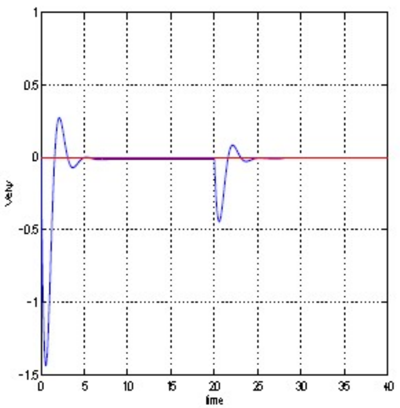
Disturbances profile 3:



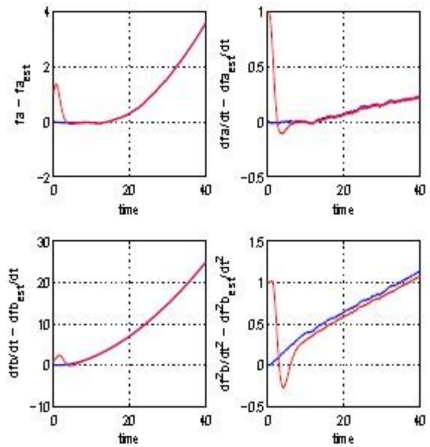
Control of x-axis velocity



Control of y-axis velocity



Control of rotational velocity

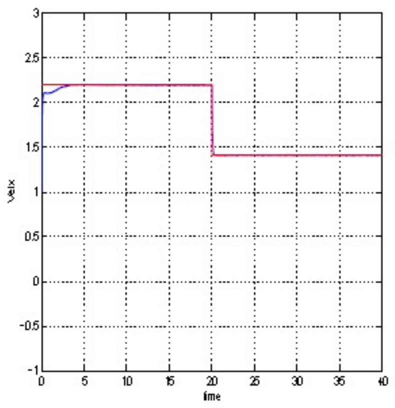


Disturbances estimation

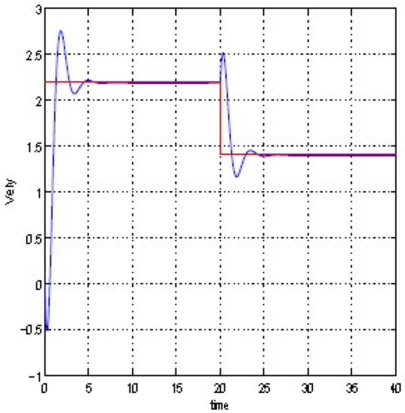
5 . Example 1: Nonlinear control and state estimation with global linearization

5.8. Simulation results

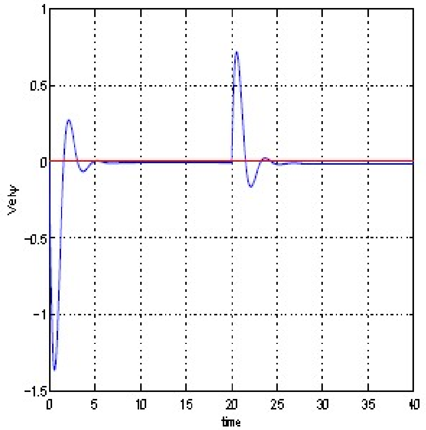
Disturbances profile 4:



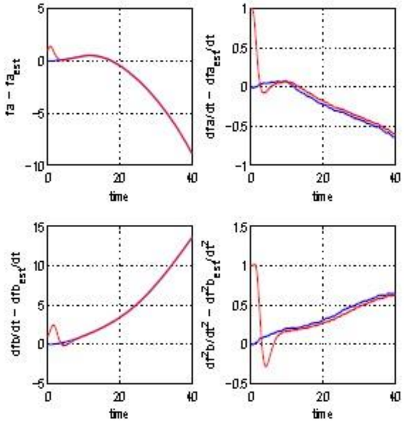
Control of x-axis velocity



Control of y-axis velocity



Control of rotational velocity



Disturbances estimation

5 . Example 1: Nonlinear control and state estimation with global linearization

5.9. Conclusions

• Two **AGV** design problems have been treated (i) **nonlinear control for autonomous navigation** (ii) **real time estimation of disturbances** due to forces or torques affecting the vehicle's motion.

• Once such disturbances have been identified with the use of a **nonlinear filtering algorithm**, that is redesigned in the form of a **disturbance observer**, it is possible to include an additional element in the vehicle's controller **that compensates for the disturbances' effects**.

• The proposed nonlinear **controller is based on differential flatness theory**. It is shown that the vehicle's model is a differentially flat one, which means that all its state variables and control inputs can be written as functions of the flat output and its derivatives.

• The transformation into the **linear canonical (Brunovsky) form** is also used to obtain an estimator of the vehicle's state vector through the processing of measurements from on-board sensors. To this end **the Derivative-free nonlinear Kalman Filter** is used.

• By redesigning the Kalman Filter algorithm in the form of a disturbance observer it is also possible to **estimate in real-time the effects of disturbance forces and torques** that are exerted on the vehicle's model and of terms representing unknown system dynamics.

• The performance of the **nonlinear AGV controller** and of the **Kalman Filter-based disturbances estimator** has been evaluated through simulation experiments.

6 . Example 2: Control and state estimation with approximate linearization

´6.1. Overview

● The article proposes a nonlinear optimal control approach for the UAV and suspended load system.

● The dynamic model of the UAV and payload system undergoes approximate linearization with the use of Taylor series expansion around a temporary operating point which recomputed at each iteration of the control method.

● For the approximately linearized model an H-infinity feedback controller is designed. The linearization procedure relies on the computation of the Jacobian matrices of the state-space model of the system.

● The control method is the solution of the optimal control problem for the nonlinear and multivariable dynamics of the UAV, under model uncertainties and external perturbations.

● For the computation of the controller's feedback gains an algebraic Riccati equation is solved at each time-step of the control method.

● The nonlinear optimal control approach achieves fast and accurate tracking for all state variables of the UAV and payload system, under moderate variations of the control inputs.

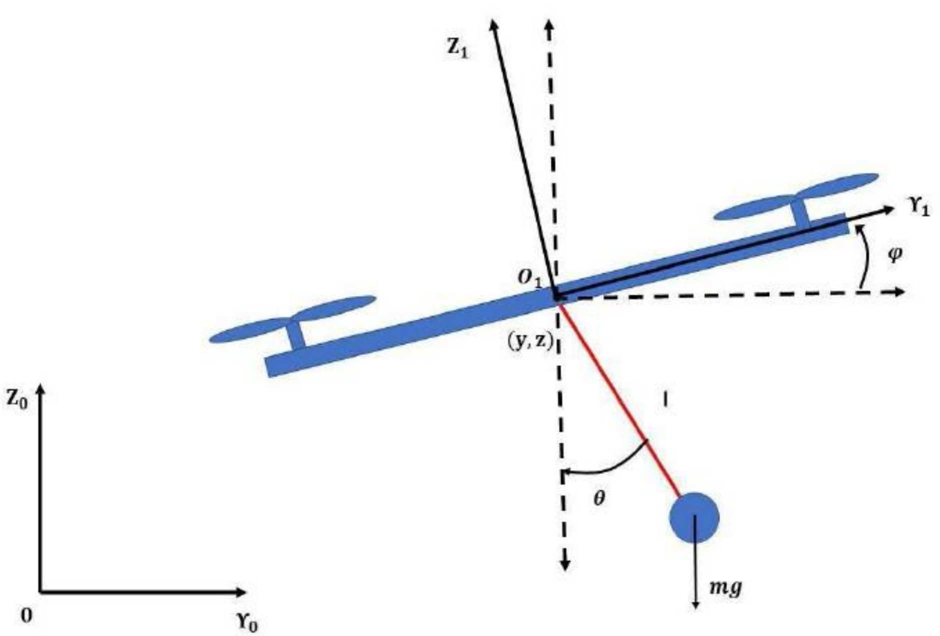
● The stability properties of the control scheme are proven through Lyapunov analysis. Finally to implement state estimation-based control the H-infinity Kalman Filter is used as a robust state estimator

6 . Example 2: Control and state estimation with approximate linearization

6.2. Dynamic model of the UAV and suspended payload system

The main variables of the dynamic model of this aerial robotic system are defined as follows: :

● φ: is the roll angle of the UAV with respect to the horizontal axis of the inertial reference frame system,
● θ: is the rotation angle of the payload wit respect to the vertical axis of the inertial reference frame,
● *l:* is the length of the string connecting the payload with the center of gravity of the UAV.



The mass of the UAV is denoted as M whereas the mass of the load is denoted as m.

Fig. 1: Reference frames for the robotic system of the UAV and suspended payload

6 . Example 2: Control and state estimation with approximate linearization

6.2. Dynamic model of the UAV and suspended payload system

● After applying the Euler-Lagrange method, the dynamic model of the UAV and of the suspended to it payload is given by the following set of differential equations

$$\begin{gathered}(M+m)\ddot{y}+ml(\ddot{\theta}cos(\theta)-\dot{\theta}^2sin(\theta))=-f\,sin(\phi)\\(M+m)(\ddot{z}+g)+ml(\ddot{\theta}sin(\theta)+\dot{\theta}^2cos(\theta))=f\,cos(\phi)\\ml\ddot{y}cos(\theta)+ml\ddot{z}sin(\theta)+ml^2\ddot{(\theta)}+mglsin(\theta)=0\\J\ddot{\phi}=\tau\end{gathered} \tag{1}$$

● The control inputs to the model are the aggregate lift force f and the torque that is generated when the motors of the UAV function at different turn speed and provide uneven power to the UAV

● The dynamic model of the UAV and of the suspended to it payload is given by the following two sets of differential equations

$$\begin{pmatrix}(M+m) & 0 & mlcos(\theta)\\0 & (M+m) & mlsin(\theta)\\mlcos(\theta) & mlsin(\theta) & -ml\end{pmatrix}\begin{pmatrix}\ddot{y}\\\ddot{z}\\\ddot{\theta}\end{pmatrix}+\begin{pmatrix}-ml\dot{\theta}^2sin(\theta)\\ml\dot{\theta}^2cos(\theta)\\mglsin(\theta)\end{pmatrix}=\begin{pmatrix}-f\,sin(\phi)\\f\,cos(\phi)-(M+m)g\\0\end{pmatrix} \tag{2}$$

$$J\ddot{\phi}=\tau \tag{3}$$

6 . Example 2: Control and state estimation with approximate linearization

6.2. Dynamic model of the UAV and suspended payload system

- By denoting $v_1 = -f sin(\phi)$, and $v_2 = f cos(\phi) - (M+m)g$

  one has $f = \{v_1^2 + [v_2 + (M+m)g]^2\}^{\frac{1}{2}}.$

- This allows also to write the state-space model as:

$$\begin{pmatrix} (M+m) & 0 & mlcos(\theta) \\ 0 & (M+m) & mlsin(\theta) \\ mlcos(\theta) & mlsin(\theta) & -ml \end{pmatrix} \begin{pmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} -ml\dot{\theta}^2 sin(\theta) \\ ml\dot{\theta}^2 cos(\theta) \\ mglsin(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (4)$$

and using the state vector $X_m = [y, z, \theta]^T$, one has also the concise form::

$$M(X_m)\ddot{X}_m + h(X_m, \dot{X}_m) = G_m v_m \quad (5)$$

where the inertia and Coriolis matrices are defined as :

$$M(X_m) = \begin{pmatrix} (M+m) & 0 & mlcos(\theta) \\ 0 & (M+m) & mlsin(\theta) \\ mlcos(\theta) & mlsin(\theta) & -ml \end{pmatrix} \quad h(x, \dot{X}) = \begin{pmatrix} -ml\dot{\theta}^2 sin(\theta) \\ ml\dot{\theta}^2 cos(\theta) \\ mglsin(\theta) \end{pmatrix} \quad G_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (6)$$

6 . Example 2: Control and state estimation with approximate linearization

6.2. Dynamic model of the UAV and suspended payload system

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- The inverse of the inertia matrix M is given by

$$M^{-1} = \frac{1}{detM} \begin{pmatrix} -(M+m)ml-(ml)^2sin^2(\theta) & (ml)^2sin(\theta)cos(\theta) & -ml(M+m)cos(\theta) \\ -(ml)^2sin(\theta)cos(\theta) & -ml(M+m)-(ml)^2cos^2(\theta) & -ml(M+m)sin(\theta) \\ -ml(M+m)cos(\theta) & -ml(M+m)sin(\theta) & (M+m)^2 \end{pmatrix}$$

where the determinant det(M) is given by $det(M) = -ml(M+m)[M+m+ml]$.

- Thus, the state-space description of the UAV with the suspended payload on it is given by

$$\ddot{X}_m = -M^{-1}h(X_m,\dot{X}_m) + M^{-1}G_m v$$
$$\ddot{\phi} = \frac{1}{J}\tau$$

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Next, one computes the product $-M^{-1}(X)h(X_m,\dot{X}_m)$

$$-M^{-1}h(X_m,\dot{X}_m) = -\frac{1}{det(M)} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

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where $a_1 = [-(M+m)ml-(ml)^2sin^2(\theta)][-ml\dot{\theta}^2sin(\theta)] + [(ml)^2sin(\theta)cos(\theta)][ml\dot{\theta}^2cos(\theta)] - [ml(M+m)cos(\theta)][mglsin(\theta)]$,

$a_2 = [-(ml)^2sin(\theta)cos(\theta)][-ml\dot{\theta}^2sin(\theta)] + [(ml)^2sin(\theta)cos(\theta)][ml\dot{\theta}^2cos(\theta)] + [-ml(M+m)cos(\theta)][ml\dot{\theta}^2cos(\theta)]$

$a_3 = [-ml(M+m)cos(\theta)][-ml\dot{\theta}^2sin(\theta)] + [-ml(M+m)sin(\theta)][ml\dot{\theta}^2cos(\theta)] + (M+m)^2[mglsin(\theta)]$

6 . Example 2: Control and state estimation with approximate linearization

6.2. Dynamic model of the UAV and suspended payload system

as well as the product $M^{-1}(X_m)G_m$

$$-M^{-1}G_m = -\frac{1}{det(M)} \cdot \begin{pmatrix} -(M+m)ml-(ml)^2 sin^2(\theta) & (ml)^2 sin(\theta)cos(\theta) \\ -(ml)^2 sin(\theta)cos(\theta) & -(M+m)ml-(ml)^2 cos^2(\theta) \\ -ml(M+m)cos(\theta) & -ml(M+m)sin(\theta) \end{pmatrix} \quad (10)$$

By defining the complete state vector as $\bar{X} = [y, \dot{y}, z, \dot{z}, \theta, \dot{\theta}, \phi, \dot{\phi}]^T$

one obtains the state-space description: $\dot{X} = F(X) + G(x)u$ (11)

where $F = [F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8]^T$ (12)

$F_1 = x_2 \quad F_3 = x_4 \quad F_5 = x_6 \quad F_7 = x_8 \quad F_8 = 0$

$$F_2 = -\frac{1}{det(M)}\{[-(M+m)ml-(ml)^2 sin^2(\theta)][-ml\dot{\theta}^2 sin(\theta)] + [(ml)^2 sin(\theta)cos(\theta)][ml\dot{\theta}^2 cos(\theta)] - [ml(M+m)cos(\theta)][mgl sin(\theta)]\}$$

$$F_4 = -\frac{1}{det(M)} \cdot \{[-(ml)^2 sin(\theta)cos(\theta)] \cdot [-ml\dot{\theta}^2 sin(\theta)] + [(ml)^2 sin(\theta)cos(\theta)] \cdot [ml\dot{\theta}^2 cos(\theta)] + [-ml(M+m)cos(\theta)] \cdot [ml\dot{\theta}^2 cos(\theta)]\}$$

$$F_6 = -\frac{1}{det(M)}\{[-ml(M+m)cos(\theta)][-ml\dot{\theta}^2 sin(\theta)] + [-ml(M+m)sin(\theta)][ml\dot{\theta}^2 cos(\theta)] + (M+m)^2[mgl sin(\theta)]\}$$

6 . Example 2: Control and state estimation with approximate linearization

6.2. Dynamic model of the UAV and suspended payload system

and

$$G = \begin{pmatrix} 0 & 0 & 0 \\ g_{11} & g_{12} & g_{13} \\ 0 & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ 0 & 0 & 0 \\ g_{31} & g_{32} & g_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (13)$$

$$G_1 = [0, g_{11}, 0, g_{21}, 0, g_{31}, 0, 0]^T$$

$$G_2 = [0, g_{12}, 0, g_{22}, 0, g_{32}, 0, 0]^T$$

$$G_3 = [0, g_{13}, 0, g_{23}, 0, g_{33}, 0, 1]^T$$

where

$$g_{11} = \frac{1}{det(M)}[-(M+m)]ml-(ml)^2 sin \quad g_{43} = \frac{1}{det(M)}$$

$$g_{21} = \frac{1}{det(M)}[-(ml)^2 sin(\theta)cos(\theta)]$$

$$g_{31} = \frac{1}{det(M)}[-ml(M+m)cos(\theta)]$$

$$g_{32} = \frac{1}{det(M)}[-ml(M+m)sin(\theta)]$$

$$g_{12} = \frac{1}{det(M)}[(ml)^2 sin(\theta)cos(\theta)]$$

$$g_{22} = \frac{1}{det(M)}[-ml(M+m)-(ml)^2 cos^2(\theta)]$$

$$g_{13} = 0, \qquad g_{23} = 0 \qquad g_{43} = \frac{1}{det(M)}$$

6 . Example 2: Control and state estimation with approximate linearization

6.3. Approximate linearization of the UAV and suspended payload system

$$\dot{X} = F(X) + G(x)u \qquad (14)$$

takes place around a **time-varying equilibrium** which is re-computed at each time instant.

This consists of the **present value of system's state vector *x*** and of the l**ast value of the control inputs vector *u*** that was applied on it.

This results into a linearized **state-space description of the form**:

$$\dot{x} = Ax + Bu + \tilde{d} \qquad (15)$$

where $\tilde{d}$ is the modelling error due to **approximate linearization** and truncation of higher-order terms in the **Taylor series expansion**, while matrices A and B are given by

$$A = \nabla_x [F(x) + G(x)u] \mid_{(x^*, u^*)} \Rightarrow$$
$$A = \nabla_x F(x) \mid_{(x^*, u^*)} + \nabla_x G_1(x) \mid_{(x^*, u^*)} u_1 + \nabla_x G_2(x) \mid_{(x^*, u^*)} u_2 + \nabla_x G_3(x) \mid_{(x^*, u^*)} u_3$$

$$B = \nabla_u [F(x) + G(x)u] \mid_{(x^*, u^*)} \Rightarrow B = G(x) \mid_{(x^*, u^*)} \qquad (16)$$

Next, the elements of the **model's Jacobian matrices** are computed:

6 . Example 2: Control and state estimation with approximate linearization

6.3. Approximate linearization of the UAV and suspended payload system

Jacobian matrix

$$\nabla_x F(x)\mid_{(x^*,u^*)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_8} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_8} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_8}{\partial x_1} & \frac{\partial F_8}{\partial x_2} & \cdots & \frac{\partial F_8}{\partial x_8} \end{pmatrix}$$

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About the first row of the Jacobian matrix $\nabla_x F(x)\mid_{(x^*,u^*)}$

$$\frac{\partial F_1}{\partial x_1}=0,\ \frac{\partial F_1}{\partial x_2}=1,\ \frac{\partial F_1}{\partial x_3}=0,\ \frac{\partial F_1}{\partial x_4}=0,\ \frac{\partial F_1}{\partial x_5}=0,\ \frac{\partial F_1}{\partial x_6}=0,\ \frac{\partial F_1}{\partial x_7}=0,\ \frac{\partial F_1}{\partial x_8}=0$$

About the second row of the Jacobian matrix $\nabla_x F(x)\mid_{(x^*,u^*)}$

$$\frac{\partial F_2}{\partial x_1}=0,\ \frac{\partial F_2}{\partial x_2}=0,\ \frac{\partial F_2}{\partial x_3}=0,\ \frac{\partial F_2}{\partial x_4}=0 \qquad \frac{\partial F_2}{\partial x_7}=0,\ \frac{\partial F_2}{\partial x_8}=0.$$

$$\begin{aligned}\frac{\partial F_2}{\partial x_5} = -\frac{1}{det(M)}\{&[-(ml)^2 2sin(x_5)cos(x_5)][-(ml)x_6^2 sin(x_5)]\\ &+[-(M+m)ml-(ml)^2 sin^2(x_5)]\cdot[-mlx_6^2 cos(x_5)]\\ &+[(ml)^2(cos^2(x_5)-sin^2(x_5))]\cdot[mlx_6^2 cos(x_5)]+[(ml)^2 sin(x_5)cos(x_5)][-mlx_6^2 sin(x_5)]\\ &-[-(ml)(M+m)sin(x_5)]\cdot[mglsin(x_5)]-[-(ml)(M+m)cos(x_5)]\cdot[mglcos(x_5)]\}.\end{aligned}$$

$$\begin{aligned}\frac{\partial F_2}{\partial x_6} = -\frac{1}{det(M)}\{&[-(M+m)(ml)-(ml)^2 sin^2(x_5)][-ml2x_6\dot{x}_6 sin(x_5)]\\ &+[(ml)^2 sin(x_5)cos(x_5)][ml2x_6\dot{x}_6 cos(x_5)]\}.\end{aligned}$$

6 . Example 2: Control and state estimation with approximate linearization

6.3. Approximate linearization of the UAV and suspended payload system

About the third row of the Jacobian matrix $\nabla_x F(x)\,|_{(x^*,u^*)}$

$$\frac{\partial F_3}{\partial x_1}=0,\ \frac{\partial F_3}{\partial x_2}=0,\ \frac{\partial F_3}{\partial x_3}=0,\ \frac{\partial F_3}{\partial x_4}=1,\quad \frac{\partial F_3}{\partial x_5}=0,\ \frac{\partial F_3}{\partial x_6}=0,\ \frac{\partial F_3}{\partial x_7}=0,\ \frac{\partial F_3}{\partial x_8}=0.$$

About the fourth row of the Jacobian matrix $\nabla_x F(x)\,|_{(x^*,u^*)}$

$$\frac{\partial F_4}{\partial x_1}=0,\ \frac{\partial F_4}{\partial x_2}=0,\ \frac{\partial F_4}{\partial x_3}=0,\ \frac{\partial F_4}{\partial x_4}=0 \qquad \frac{\partial F_4}{\partial x_7}=0,\ \frac{\partial F_4}{\partial x_8}=0.$$

$$\begin{aligned}\frac{\partial F_4}{\partial x_5}=-\frac{1}{det(M)}\{&[-(ml)^2(cos^2(x_5)-sin^2(x_5))][-mlx_6^2sin(x_5)]+[-(ml)^2sin(x_5)cos(x_5)][-mlx_6^2cos(x_5)]\\ &+[(ml)^2(cos^2(x_5)-sin^2(x_5))][mlx_6^2cos(x_5)]+[(ml)^2sin(x_5)cos(x_5)][-(ml)x_6^2sin(x_5)]\\ &+[ml(M+m)sin(x_5)][mlx_6^2cos(x_5)]+[-ml(M+m)cos(x_5)][-mlx_6^2sin(x_5)]\}\end{aligned}$$

$$\begin{aligned}\frac{\partial F_4}{\partial x_5}=-\frac{1}{det(M)}\{&[-(ml)^2(cos^2(x_5)-sin^2(x_5))][-mlx_6^2sin(x_5)]+[-(ml)^2sin(x_5)cos(x_5)][-mlx_6^2cos(x_5)]\\ &+[(ml)^2(cos^2(x_5)-sin^2(x_5))][mlx_6^2cos(x_5)]+[(ml)^2sin(x_5)cos(x_5)][-(ml)x_6^2sin(x_5)]\\ &+[ml(M+m)sin(x_5)][mlx_6^2cos(x_5)]+[-ml(M+m)cos(x_5)][-mlx_6^2sin(x_5)]\}\end{aligned}$$

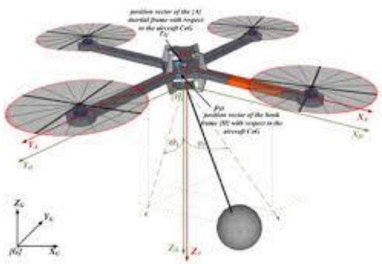
About the fifth row of the Jacobian matrix $\nabla_x F(x)\,|_{(x^*,u^*)}$

$$\frac{\partial F_5}{\partial x_1}=0,\ \frac{\partial F_5}{\partial x_2}=0,\ \frac{\partial F_5}{\partial x_3}=0,\ \frac{\partial F_5}{\partial x_4}=0,\quad \frac{\partial F_5}{\partial x_5}=0,\ \frac{\partial F_5}{\partial x_6}=1,\ \frac{\partial F_5}{\partial x_7}=0,\ \frac{\partial F_5}{\partial x_8}=0$$

6 . Example 2: Control and state estimation with approximate linearization

6.3. Approximate linearization of the UAV and suspended payload system

About the sixth row of the Jacobian matrix $\nabla_x F(x)\,|_{(x^*,u^*)}$



$$\frac{\partial F_6}{\partial x_1}=0,\ \frac{\partial F_6}{\partial x_2}=0,\ \frac{\partial F_6}{\partial x_3}=0,\ \frac{\partial F_6}{\partial x_4}=0 \qquad \frac{\partial F_6}{\partial x_7}=0,\ \frac{\partial F_6}{\partial x_8}=0.$$

$$\frac{\partial F_6}{\partial x_5}=-\frac{1}{det(M)}\{[ml(M+m)sin(x_5)][-mlx_6^2sin(x_5)]$$
$$[-ml(M+m)cos(x_5)][-mlx_6^2cos(x_5)]+[-ml(M+m)cos(x_5)][mlx_6^2cos(x_5)]+$$
$$[-ml(M+m)sin(x_5)][-mlx_6^2sin(x_5)]+(M+m)^2[mglcos(x_5)]\}$$

$$\frac{\partial F_6}{\partial x_6}=-\frac{1}{det(M)}\{[-ml(M+m)cos(x_5)][-ml2x_6\dot{x}_6sin(x_5)]$$
$$+[-ml(M+m)sin(x_5)][ml2x_6\dot{x}_6cos(x_5)]\}$$

About the seventh row of the Jacobian matrix $\nabla_x F(x)\,|_{(x^*,u^*)}$

$$\frac{\partial F_7}{\partial x_1}=0,\ \frac{\partial F_7}{\partial x_2}=0,\ \frac{\partial F_7}{\partial x_3}=0,\ \frac{\partial F_7}{\partial x_4}=0,\ \frac{\partial F_7}{\partial x_5}=0,\ \frac{\partial F_7}{\partial x_6}=0,\ \frac{\partial F_7}{\partial x_7}=0,\ \frac{\partial F_7}{\partial x_8}=1.$$

About the eight row of the Jacobian matrix $\nabla_x F(x)\,|_{(x^*,u^*)}$

$$\frac{\partial F_8}{\partial x_1}=0,\ \frac{\partial F_8}{\partial x_2}=0,\ \frac{\partial F_8}{\partial x_3}=0,\ \frac{\partial F_8}{\partial x_4}=0,\ \frac{\partial F_8}{\partial x_5}=0,\ \frac{\partial F_8}{\partial x_6}=0,\ \frac{\partial F_8}{\partial x_7}=0,\ \frac{\partial F_8}{\partial x_8}=1.$$

6 . Example 2: Control and state estimation with approximate linearization

6.3. Approximate linearization of the UAV and suspended payload system

The Jacobian matrix $\nabla_x G_1(x)\mid_{(x^*,u^*)}$ of the robotic system is computed as follows:

(18)

$$\nabla_x G_1(x)\mid_{(x^*,v^*)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g_{11}}{\partial x_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g_{21}}{\partial x_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g_{31}}{\partial x_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\frac{\partial g_{11}}{\partial x_5} = -(ml)^2 2sin(x_5)cos(x_5)$,

$$\frac{\partial g_{21}}{\partial x_5} = -(ml)^2(cos^2(x_5) - sin^2(x_5))$$

$$\frac{\partial g_{31}}{\partial x_5} = ml(M+m)sin(x_5).$$

The Jacobian matrix $\nabla_x G_2(x)\mid_{(x^*,u^*)}$ of the robotic system is computed as follows:

(19)

$$\nabla_x G_2(x)\mid_{(x^*,u^*)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g_{12}}{\partial x_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g_{22}}{\partial x_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g_{32}}{\partial x_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\frac{\partial g_{12}}{\partial x_5} = ml(cos^2(x_5) - sin^2(x_5))$,

$$\frac{\partial g_{22}}{\partial x_5} = (ml)^2 2cos(x_5)sin(x_5)$$

$$\frac{\partial g_{32}}{\partial x_5} = -ml(M+m)cos(x_5).$$

The Jacobian matrix $\nabla_x G_3(x)\mid_{(x^*,u^*)}$ of the robotic system is computed as follows:

$$\nabla_x G_3(x)\mid_{(x^*,v^*)}= 0_{8\times 8}.$$

(20)

6 . Example 2: Control and state estimation with approximate linearization

6.4. Design of an H-infinity controller for the UAV and suspended payload system

As explained, the system's dynamic model undergoes **linearization** round its present operating point (x^*,u^*), where x^* is the present value of the UAV and payload's state vector and u^* is the last value of the control input vector that was applied on it. Thus one arrives at the **approximately linearized description** of the system:

$$\dot{x} = Ax + Bu + \tilde{d} \quad (21)$$

where d_1 is the linearization error due to truncation of higher-order terms in the **Taylor series expansion** and

$$A = \nabla_x[f(x) + g(x)u] \mid_{(x^*,u^*)} \Rightarrow A = \nabla_x[f(x) \mid_{(x^*,u^*)} + \nabla_x[g(x)u \mid_{(x^*,u^*)}] \quad (22)$$

In a similar manner, one has that

$$B = \nabla_u[f(x) + g(x)u] \mid_{(x^*,u^*)} \Rightarrow B = g(x) \mid_{(x^*,u^*)} \quad (23)$$

After **linearization** round its current operating point the system's **model** is written as

$$\dot{x} = Ax + Bu + d_1 \quad (24)$$

Parameter d_1 stands for the **linearization error** in the system's state-space model

At every time instant the control input u^* is assumed to differ from the control input u appearing in (21) by an amount equal to Δu, that is $u^* = u + \Delta u$

$$\dot{x}_d = Ax_d + Bu^* + d_2 \quad (25)$$

6 . Example 2: Control and state estimation with approximate linearization

6.4. Design of an H-infinity controller for the UAV and suspended payload system

The initial model of the UAV and suspended payload is assumed to be in the form

$$\dot{x} = f(x,u) \quad x \in R^n, \ u \in R^m$$

where the **linearization point (temporary equilibrium)** is defined by the present value of the system's state vector and the last value of the control inputs vector exerted on it

$$(x^*, u^*) = (x(t), u(t - T_s)).$$

The **linearized equivalent of the system** is described by

$$\dot{x} = Ax + Bu + Ld \quad x \in R^n, \ u \in R^m, \ d \in R^q \qquad (26)$$

where matrices A and B are obtained from the **computation of the Jacobians**

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \Bigg|_{(x^*, u^*)} \qquad B = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{pmatrix} \Bigg|_{(x^*, u^*)}$$

and vector d denotes disturbance terms due to linearization errors.

$$\begin{aligned} \dot{x} &= Ax + Bu + Ld \\ y &= Cx \end{aligned}$$

6 . Example 2: Control and state estimation with approximate linearization

6.4. Design of an H-infinity controller for the UAV and suspended payload system

The dynamics of the system of Eq. 24 can be also written in the form

$$\dot{x} = Ax + Bu + Bu^* - Bu^* + d_1 \quad (27)$$

and by denoting $d_3 = -Bu^* + d_1$ as an **aggregate disturbance** term one obtains

$$\dot{x} = Ax + Bu + Bu^* + d_3 \quad (28)$$

By subtracting Eq. 25 from Eq. 28 one has

$$\dot{x} - \dot{x}_d = A(x - x_d) + Bu + d_3 - d_2 \quad (29)$$

By denoting the tracking error as $e = x - x_d$ and the aggregate disturbance term as $\tilde{d} = d_3 - d_2$ the **tracking error dynamics** becomes

$$\dot{e} = Ae + Bu + \tilde{d} \quad (30)$$

6 . Example 2: Control and state estimation with approximate linearization

6.4. Design of an H-infinity controller for the UAV and suspended payload system

The problem of **disturbance rejection** for the linearized model that is described by

$$\dot{x} = Ax + Bu + Ld$$
$$y = Cx$$

where $x \in R^n$, $u \in R^m$, $d \in R^q$ and $y \in R^p$ cannot be handled efficiently if the classical LQR control scheme is applied. This is because of the existence of the perturbation term d.

In the $H\infty$ control approach, a **feedback control scheme** is designed for **trajectory tracking** by the system's state vector and simultaneous disturbance rejection, considering that the disturbance affects the system in the worst possible manner

The disturbances' effect are incorporated in the following **quadratic cost function**

$$J(t) = \frac{1}{2}\int_0^T [y^T(t)y(t) + \\ + ru^T(t)u(t) - \rho^2 d^T(t)d(t)]dt, \quad r, \rho > 0 \qquad (31)$$

The coefficient r determines **the penalization of the control input** and the weight coefficient ρ determines the **reward of the disturbances**' effects.

6 . Example 2: Control and state estimation with approximate linearization

6.4. Design of an H-infinity controller for the UAV and suspended payload system

Then, the **optimal feedback control** law is given by

$$u(t) = -Kx(t) \qquad (32)$$

with

$$K = \frac{1}{r}B^T P$$

where ***P* is a positive semi-definite symmetric matrix** which is obtained from the solution of the **Riccati equation**

$$A^T P + PA + Q - P\left(\frac{1}{r}BB^T - \frac{1}{2\rho^2}LL^T\right)P = 0 \qquad (33)$$

where Q is also a positive definite symmetric matrix.

The parameter ρ in Eq. 33 is an **indication of the closed-loop system robustness**. If the values of $\rho > 0$ are excessively decreased with respect to r, then the solution of the Riccati equation is no longer a positive definite matrix. Consequently, there is a lower bound ρ_{min} of for which the H-infinity control problem has a solution.

6 . Example 2: Control and state estimation with approximate linearization

6.5. Stability analysis for the UAV and suspended payload system

The **tracking error dynamics** for the UAV and suspended payload system is written in the form

$$\dot{e} = Ae + Bu + L\tilde{d} \qquad (34)$$

where in the **case of the considered rotary pendulum** $L = I \in R^8$ with I being the identity matrix. The following **Lyapunov function** is considered

$$V = \frac{1}{2}e^T P e \qquad (35)$$

where $e = x - x_d$ Is the state vector's tracking error

$$\dot{V} = \frac{1}{2}\dot{e}^T P e + \frac{1}{2}e^T P \dot{e} \Rightarrow$$
$$\dot{V} = \frac{1}{2}[Ae + Bu + L\tilde{d}]^T P + \frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow$$

$$\dot{V} = \frac{1}{2}[e^T A^T + u^T B^T + \tilde{d}^T L^T]Pe +$$
$$+\frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow$$

$$\dot{V} = \frac{1}{2}e^T A^T P e + \frac{1}{2}u^T B^T P e + \frac{1}{2}\tilde{d}^T L^T P e +$$
$$\frac{1}{2}e^T P A e + \frac{1}{2}e^T P B u + \frac{1}{2}e^T P L \tilde{d}$$

6 . Example 2: Control and state estimation with approximate linearization

6.5. Stability analysis for the UAV and suspended payload system

The previous equation is rewritten as

$$\dot{V} = \frac{1}{2}e^T(A^T P + PA)e + (\frac{1}{2}u^T B^T Pe + \frac{1}{2}e^T PBu) +$$
$$+(\frac{1}{2}\tilde{d}^T L^T Pe + \frac{1}{2}e^T PL\tilde{d})$$

Assumption*: For given positive definite matrix Q and coefficients* r and ρ there exists a positive definite matrix P, which is the solution of the following matrix equation

$$A^T P + PA = -Q + P(\frac{2}{r}BB^T - \frac{1}{\rho^2}LL^T)P \qquad (36)$$

Moreover, the following **feedback control law** is applied to the PEM fuel cells model

$$u = -\frac{1}{r}B^T Pe \qquad (37)$$

By substituting Eq. 37 and Eq. 36 one obtains

$$\dot{V} = \frac{1}{2}e^T[-Q + P(\frac{1}{r}BB^T - \frac{1}{2\rho^2}LL^T)P]e +$$
$$+e^T PB(-\frac{1}{r}B^T Pe + e^T PL\tilde{d} \Rightarrow$$

6 . Example 2: Control and state estimation with approximate linearization

6.5. Stability analysis for the UAV and suspended payload system

Continuing with computations one obtains

$$\dot{V} = -\frac{1}{2}e^TQe + \left(\frac{1}{r}PBB^TPe - \frac{1}{2\rho^2}e^TPLL^T\right)Pe$$
$$-\frac{1}{r}e^TPBB^TPe + e^TPL\tilde{d}$$

which next gives

$$\dot{V} = -\frac{1}{2}e^TQe - \frac{1}{2\rho^2}e^TPLL^TPe + e^TPL\tilde{d}$$

or equivalently

$$\dot{V} = -\frac{1}{2}e^TQe - \frac{1}{2\rho^2}e^TPLL^TPe +$$
$$+\frac{1}{2}e^TPL\tilde{d} + \frac{1}{2}\tilde{d}^TL^TPe \qquad (38)$$

Lemma: The following inequality holds

$$\frac{1}{2}e^TL\tilde{d} + \frac{1}{2}\tilde{d}L^TPe - \frac{1}{2\rho^2}e^TPLL^TPe \leq \frac{1}{2}\rho^2\tilde{d}^T\tilde{d}$$

6 . Example 2: Control and state estimation with approximate linearization

6.5. Stability analysis for the UAV and suspended payload system

Proof : *The binomial* $(\rho\alpha-\frac{1}{\rho}b)^2$ is considered. Expanding the left part of the above inequality one gets

$$\rho^2a^2+\frac{1}{\rho^2}b^2-2ab\geq 0\Rightarrow\frac{1}{2}\rho^2a^2+\frac{1}{2\rho^2}b^2-ab\geq 0\Rightarrow$$
$$ab-\frac{1}{2\rho^2}b^2\leq\frac{1}{2}\rho^2a^2\Rightarrow\frac{1}{2}ab+\frac{1}{2}ab-\frac{1}{2\rho^2}b^2\leq\frac{1}{2}\rho^2a^2$$

The following substitutions are carried out: $a=\tilde{d}$ and $b=e^TPL$ and the previous relation becomes

$$\frac{1}{2}\tilde{d}^TL^TPe+\frac{1}{2}e^TPL\tilde{d}-\frac{1}{2\rho^2}e^TPLL^TPe\leq\frac{1}{2}\rho^2\tilde{d}^T\tilde{d} \quad (39)$$

Eq. (39) is substituted in Eq. (38) and the inequality is enforced, thus giving

$$\dot{V}\leq-\frac{1}{2}e^TQe+\frac{1}{2}\rho^2\tilde{d}^T\tilde{d} \quad (40)$$

Eq. (40) shows that the **H-infinity tracking performance criterion** is satisfied.

The integration of $\dot{V}$ from 0 to T gives

$$\int_0^T\dot{V}(t)dt\leq-\frac{1}{2}\int_0^T||e||_Q^2dt+\frac{1}{2}\rho^2\int_0^T||\tilde{d}||^2dt\Rightarrow$$
$$2V(T)+\int_0^T||e||_Q^2dt\leq 2V(0)+\rho^2\int_0^T||\tilde{d}||^2dt$$

6 . Example 2: Control and state estimation with approximate linearization

6.5. Stability analysis for the UAV and suspended payload system

Moreover, if there exists a positive constant $M_d > 0$ such that

$$\int_0^{\infty} ||\tilde{d}||^2 dt \leq M_d$$

then one gets

$$\int_0^{\infty} ||e||_Q^2 dt \leq 2V(0) + \rho^2 M_d$$

Thus, the integral $\int_0^{\infty} ||e||_Q^2 dt$ is bounded.

Moreover, V(T) is bounded and from the definition of the Lyapunov function V it becomes clear that **e(t) will be also bounded** since

$$e(t) \in \Omega_e = \{e | e^T P e \leq 2V(0) + \rho^2 M_d\}.$$

According to the above and with the use of **Barbalat's Lemma** one obtains:

$$lim_{t \to \infty} e(t) = 0. \quad (41)$$

6 . Example 2: Control and state estimation with approximate linearization

6.6. Stability analysis for the UAV and suspended payload system

- The control loop has to be implemented with the use of information provided by a **small number of measurements** of the state variables of UAV and suspended payload system

- To reconstruct the missing information about the state vector of the pendulum's model it is proposed to **use a filter** and based on it to apply state **estimation-based control** .

- The **recursion of the H-infinity Kalman Filter**, for the UAV and suspended payload system, can be formulated in terms of a measurement update and a time update part

Measurement update

$$\begin{aligned} D(k) &= [I - \theta W(k)P^-(k) + C^T(k)R(k)^{-1}C(k)P^-(k)]^{-1} \\ K(k) &= P^-(k)D(k)C^T(k)R(k)^{-1} \\ \hat{x}(k) &= \hat{x}^-(k) + K(k)[y(k) - C\hat{x}^-(k)] \end{aligned} \quad (42)$$

Time update

$$\begin{aligned} \hat{x}^-(k+1) &= A(k)x(k) + B(k)u(k) \\ P^-(k+1) &= A(k)P^-(k)D(k)A^T(k) + Q(k) \end{aligned} \quad (43)$$

where it is assumed that parameter θ is sufficiently small to assure that the **covariance matrix**

$$P^-(k)^{-1} - \theta W(k) + C^T(k)R(k)^{-1}C(k)$$

Is **positive definite**

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

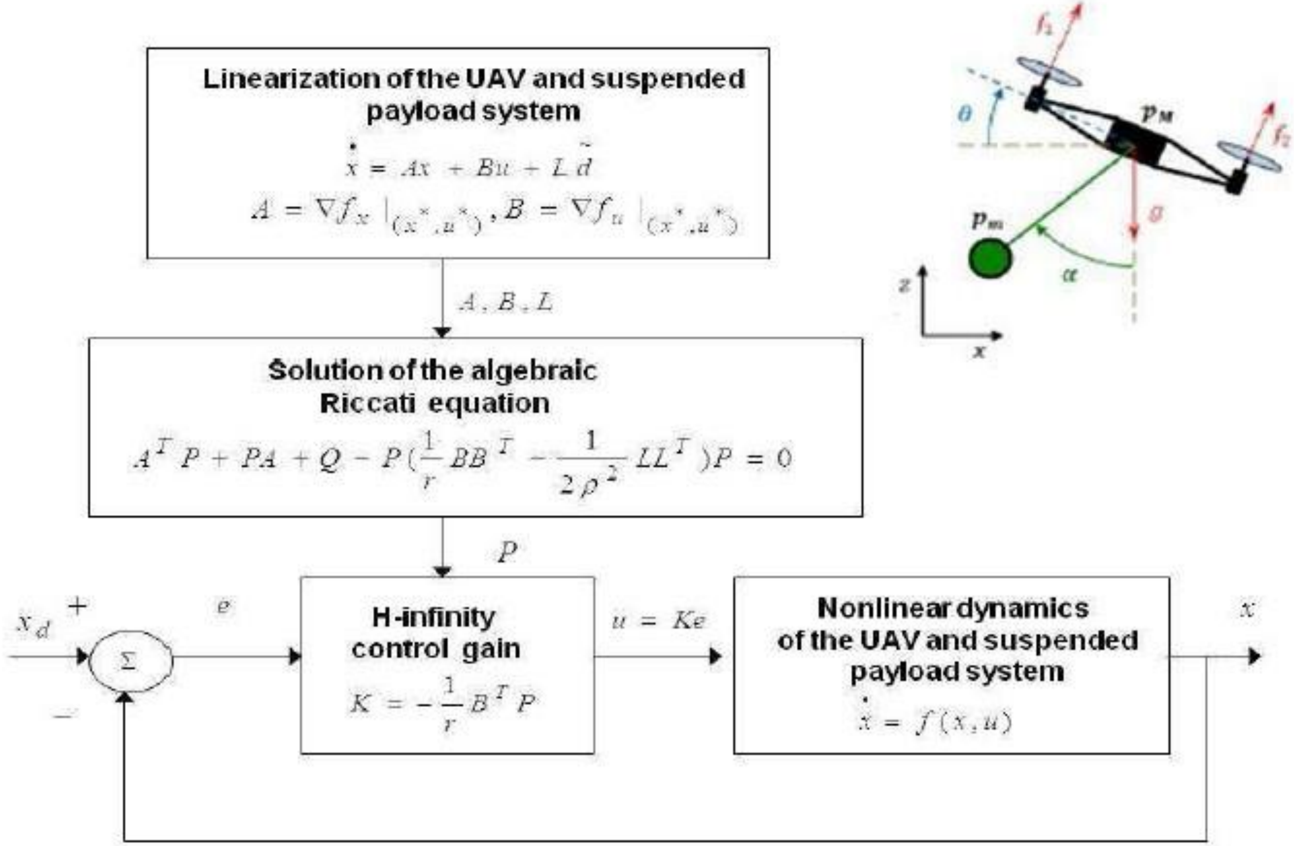


Fig. 2 Diagram of the nonlinear optimal control for the UAV and suspended payload

With the use of the proposed H-infinity control method, fast and accurate tracking of the reference setpoints of the UAV and suspended payload system was achieved

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 1

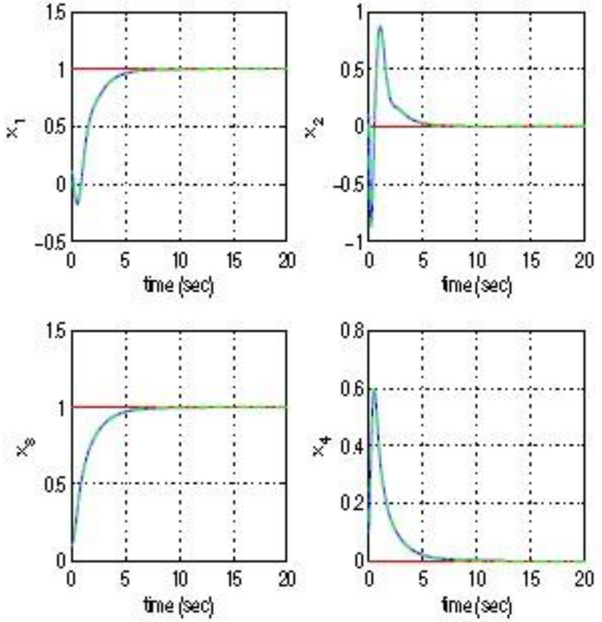


Fig. 3(a) convergence of state variables x1 (y-axis position of the UAV), x2 (y-axis velocity of the UAV), x3 (z-axis position of the UAV) and x4 (z-axis velocity of the UAV) to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value)

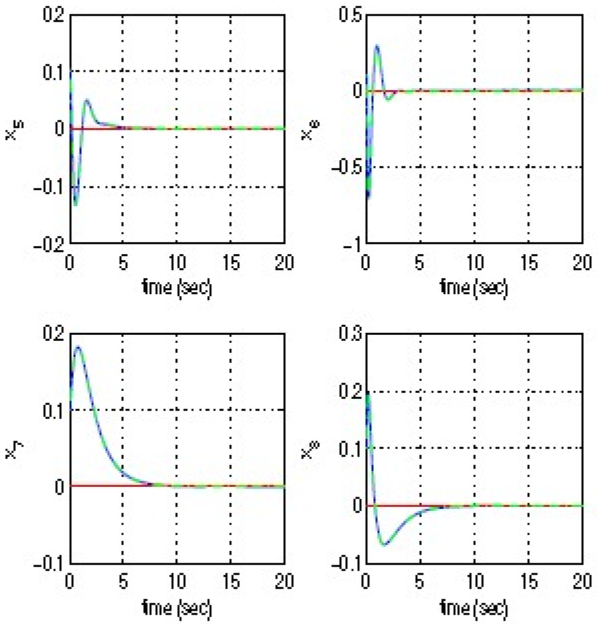
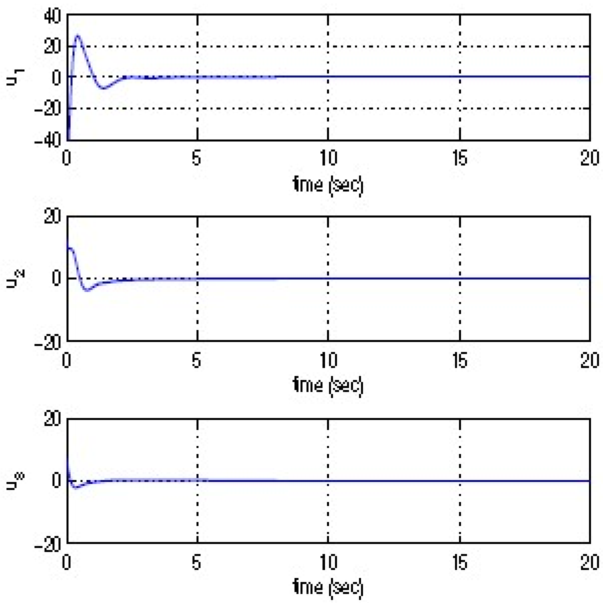


Fig 3(b) convergence of state variables x5 (rotation angle of the payload), x6 (rotational speed of the payload), x7 (roll angle of the UAV) and x8 (roll angular speed of the UAV) to their reference setpoints

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 1



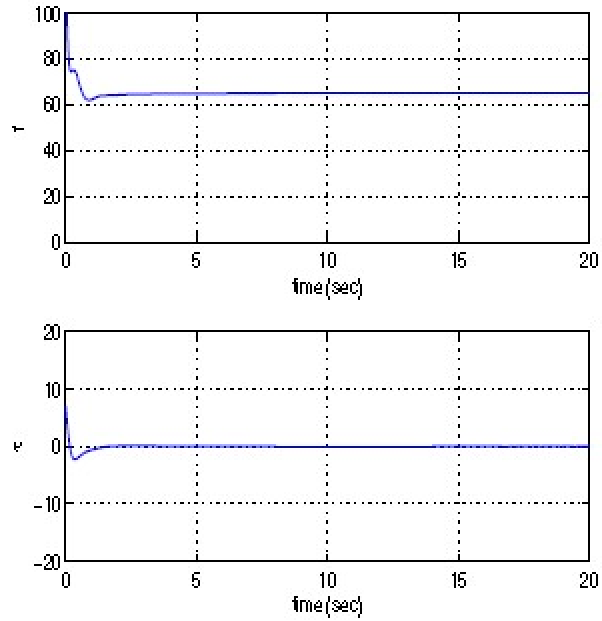


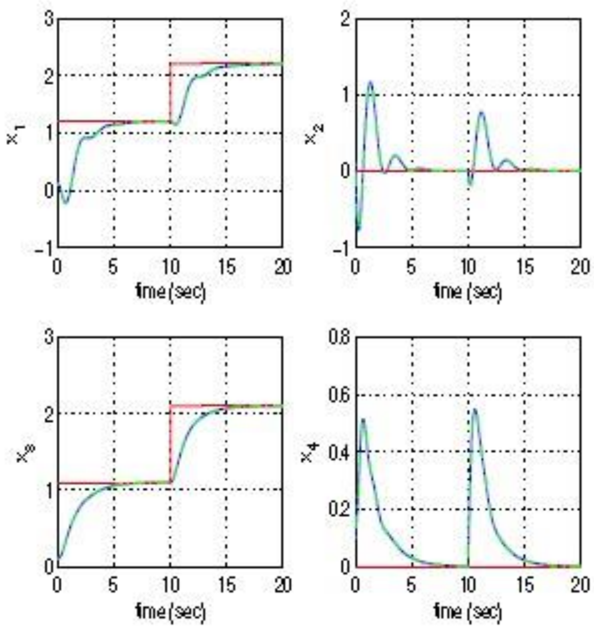
Fig. 4(a) control inputs to the UAV u_i i=1,2,3 computed through the solution of the nonlinear optimal control problem

Fig, 4(b) control inputs f (lift force of the UAV's motors) and (torque generated in aggregate by the motors of the UAV)

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 2



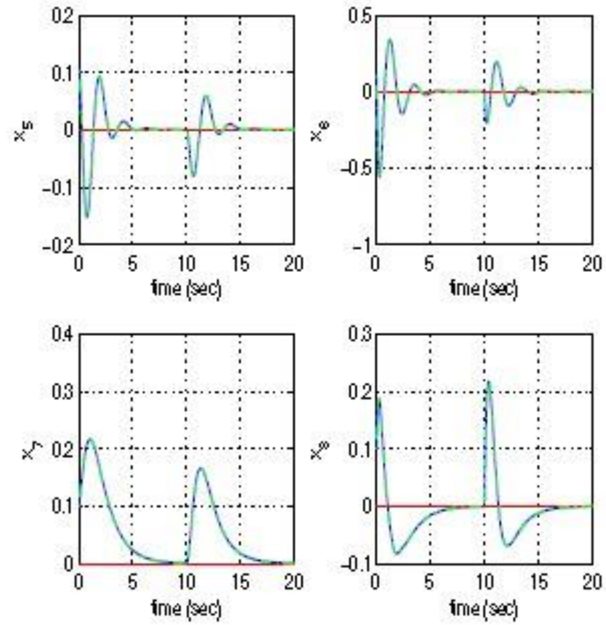


Fig. 5(a) convergence of state variables x1 (y-axis position of the UAV), x2 (y-axis velocity of the UAV), x3 (z-axis position of the UAV) and x4 (z-axis velocity of the UAV) to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value)

Fig 5(b) convergence of state variables x5 (rotation angle of the payload), x6 (rotational speed of the payload), x7 (roll angle of the UAV) and x8 (roll angular speed of the UAV) to their reference setpoints

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 2

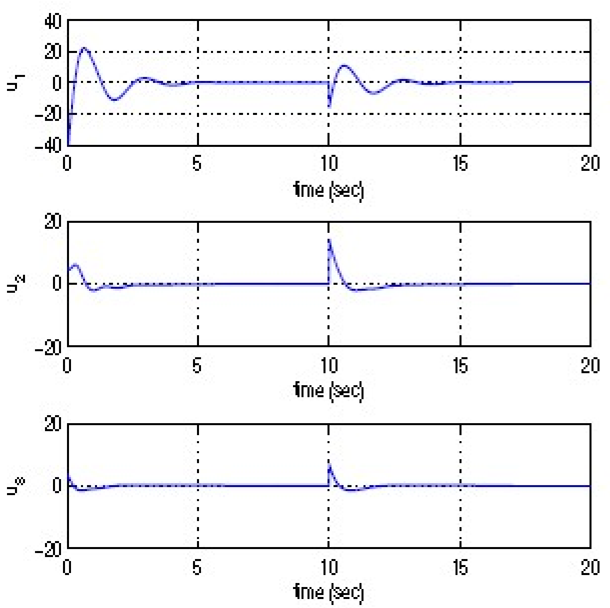
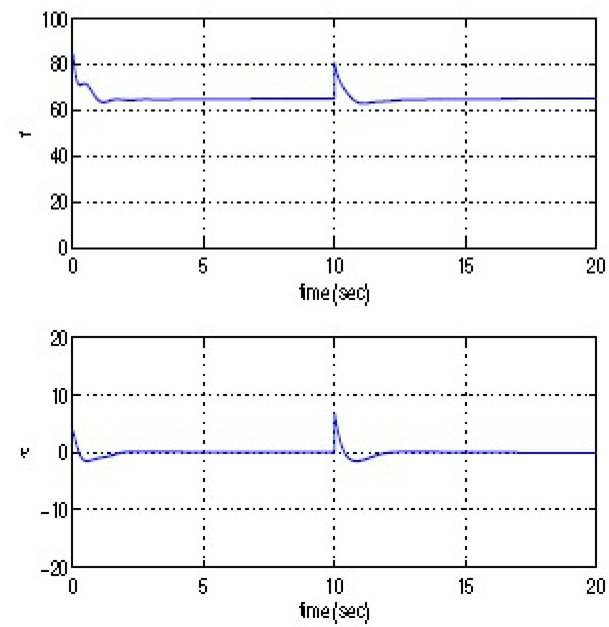


Fig. 6(a) control inputs to the UAV u_i i=1,2,3 computed through the solution of the nonlinear optimal control problem



Fig, 6(b) control inputs f (lift force of the UAV's motors) and (torque generated in aggregate by the motors of the UAV)

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 3

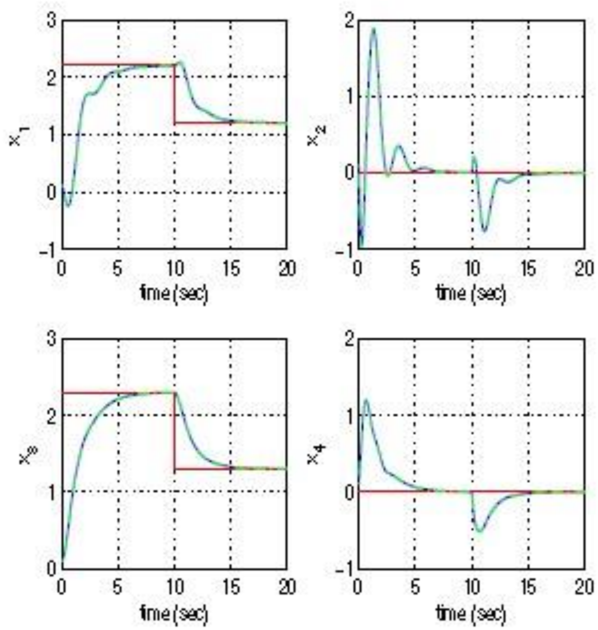


Fig. 7a) convergence of state variables x1 (y-axis position of the UAV), x2 (y-axis velocity of the UAV), x3 (z-axis position of the the UAV) and x4 (z-axis velocity of the UAV) to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value)

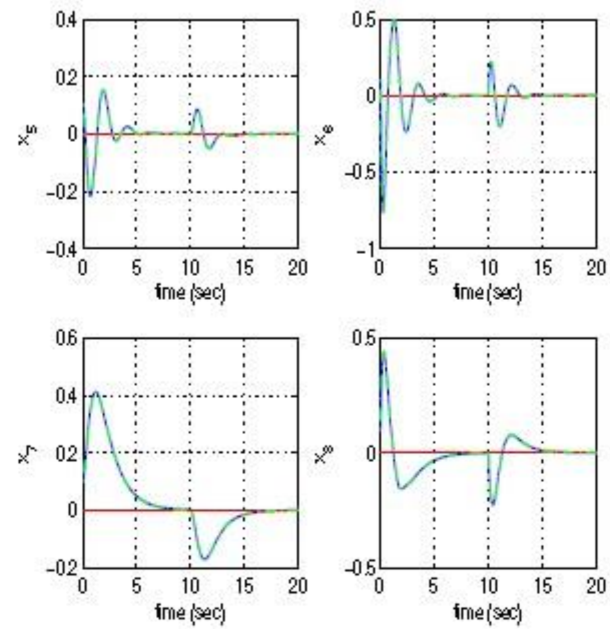
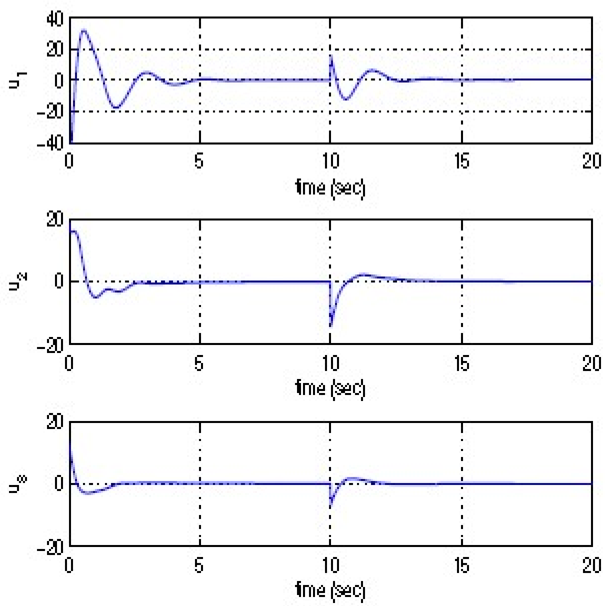


Fig 7(b) convergence of state variables x5 (rotation angle of the payload), x6 (rotational speed of the payload), x7 (roll angle of the UAV) and x8 (roll angular speed of the UAV) to their reference setpoints

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 3



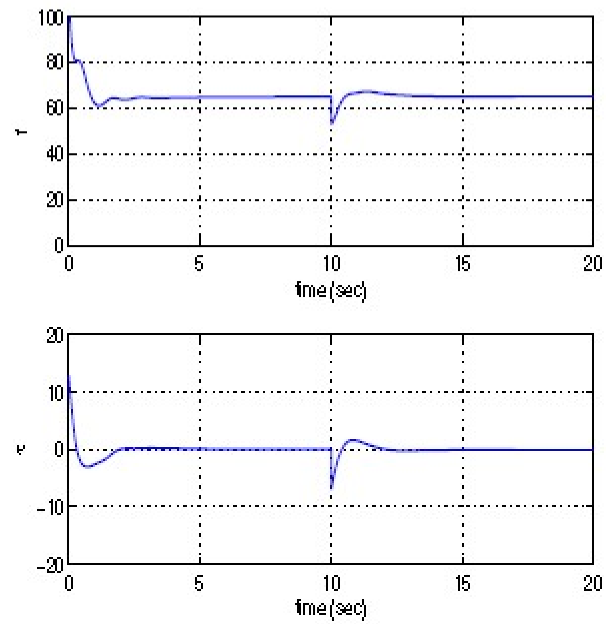


Fig. 8(a) control inputs to the UAV u_i i=1,2,3 computed through the solution of the nonlinear optimal control problem

Fig, 8(b) control inputs f (lift force of the UAV's motors) and (torque generated in aggregate by the motors of the UAV)

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 4

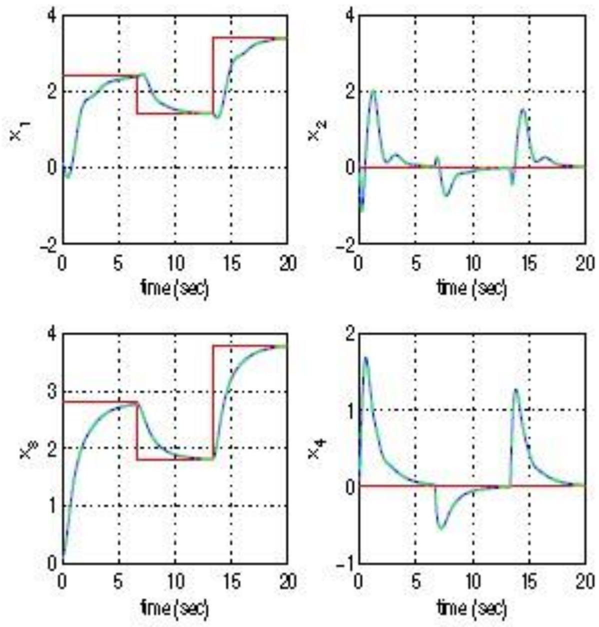


Fig. 9(a) convergence of state variables x1 (y-axis position of the UAV), x2 (y-axis velocity of the UAV), x3 (z-axis position of the the UAV) and x4 (z-axis velocity of the UAV) to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value)

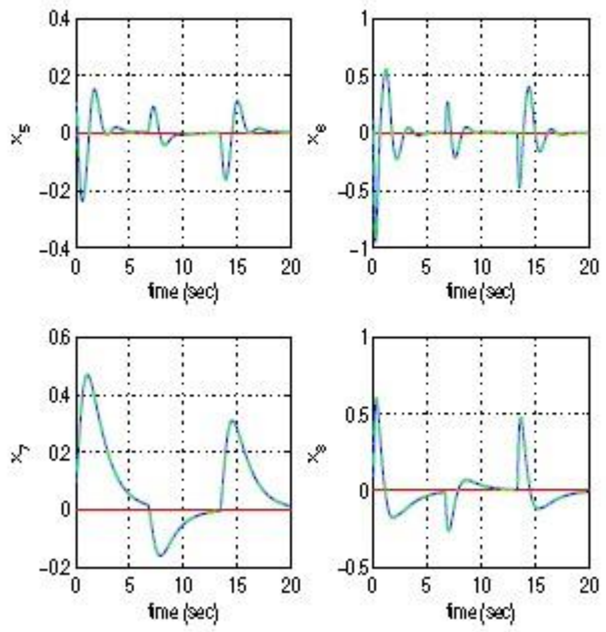


Fig 9(b) convergence of state variables x5 (rotation angle of the payload), x6 (rotational speed of the payload), x7 (roll angle of the UAV) and x8 (roll angular speed of the UAV) to their reference setpoints

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 4

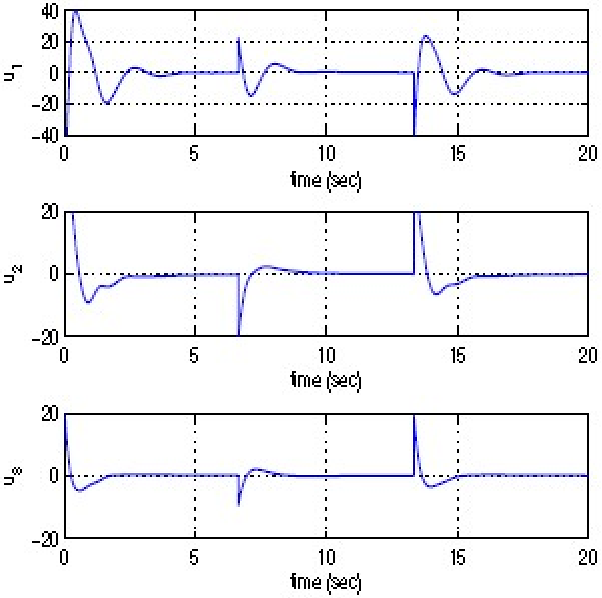
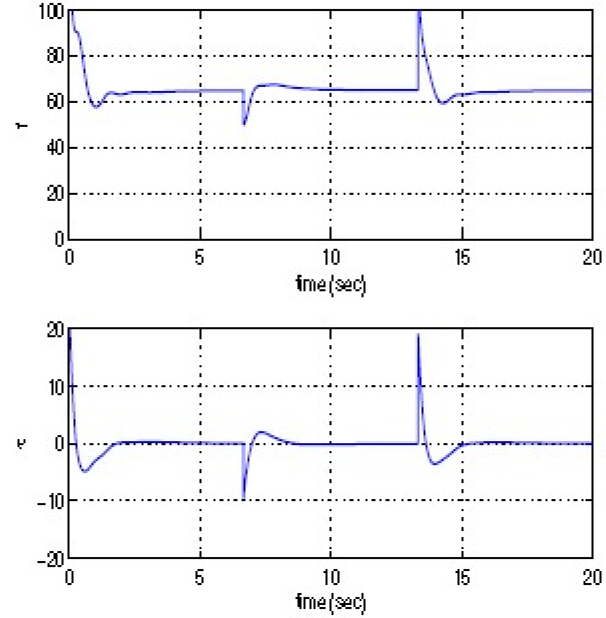


Fig. 10(a) control inputs to the UAV u_i i=1,2,3 computed through the solution of the nonlinear optimal control problem

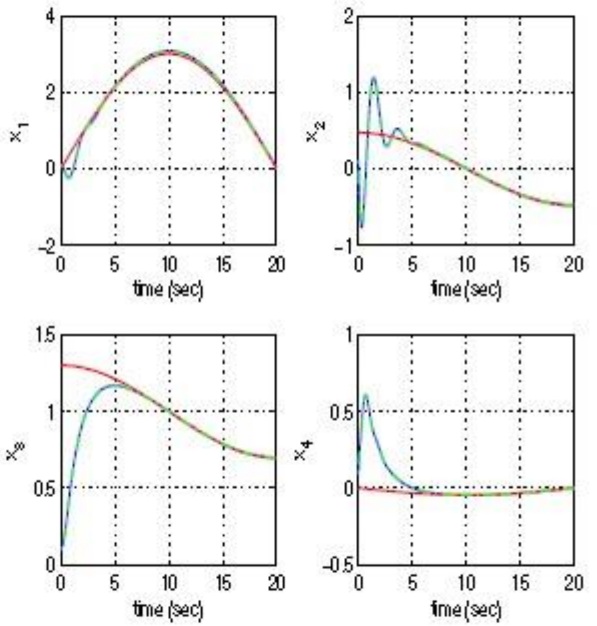


Fig, 10(b) control inputs f (lift force of the UAV's motors) and (torque generated in aggregate by the motors of the UAV)

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 5



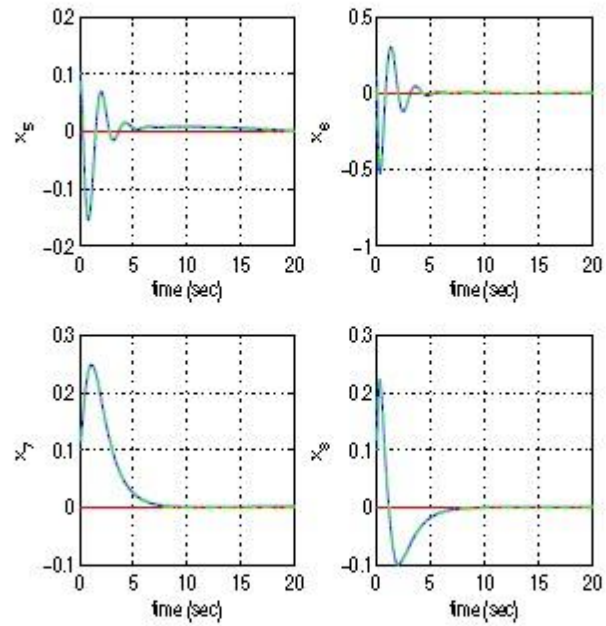


Fig. 11(a) convergence of state variables x1 (y-axis position of the UAV), x2 (y-axis velocity of the UAV), x3 (z-axis position of the UAV) and x4 (z-axis velocity of the UAV) to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value)

Fig 11(b) convergence of state variables x5 (rotation angle of the payload), x6 (rotational speed of the payload), x7 (roll angle of the UAV) and x8 (roll angular speed of the UAV) to their reference setpoints

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 5

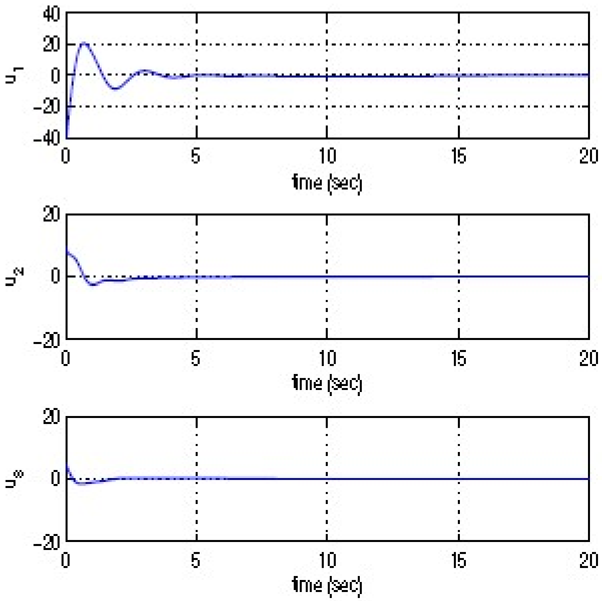
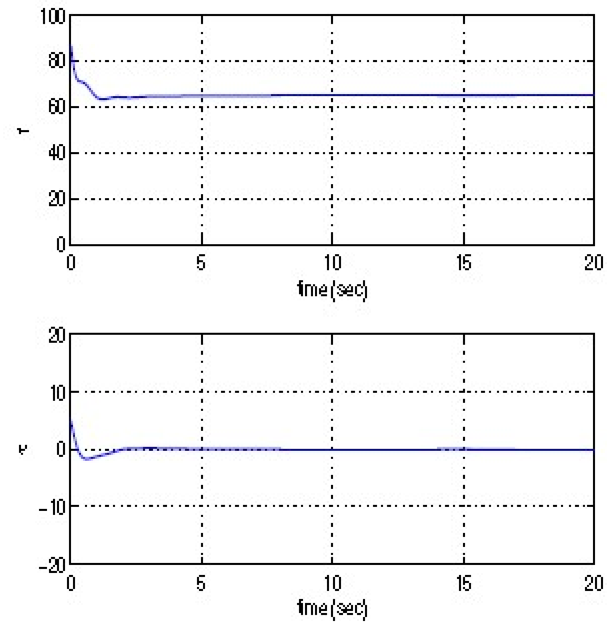


Fig. 12(a) control inputs to the UAV u_i i=1,2,3 computed through the solution of the nonlinear optimal control problem



Fig, 12(b) control inputs f (lift force of the UAV's motors) and (torque generated in aggregate by the motors of the UAV)

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 6

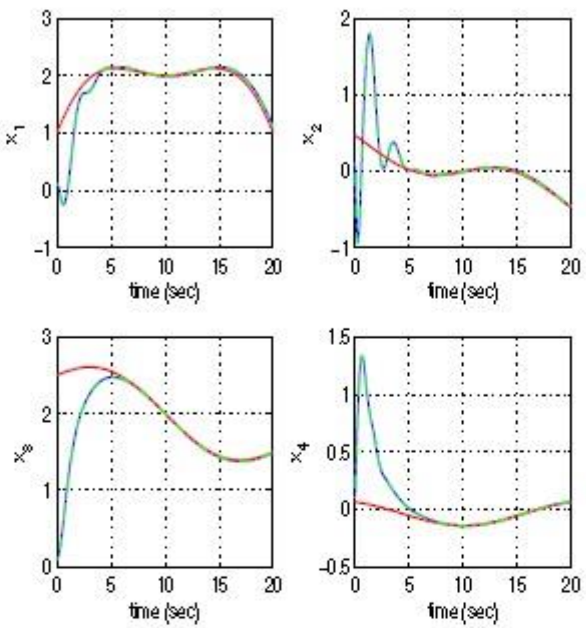


Fig. 13(a) convergence of state variables x1 (y-axis position of the UAV), x2 (y-axis velocity of the UAV), x3 (z-axis position of the UAV) and x4 (z-axis velocity of the UAV) to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value)

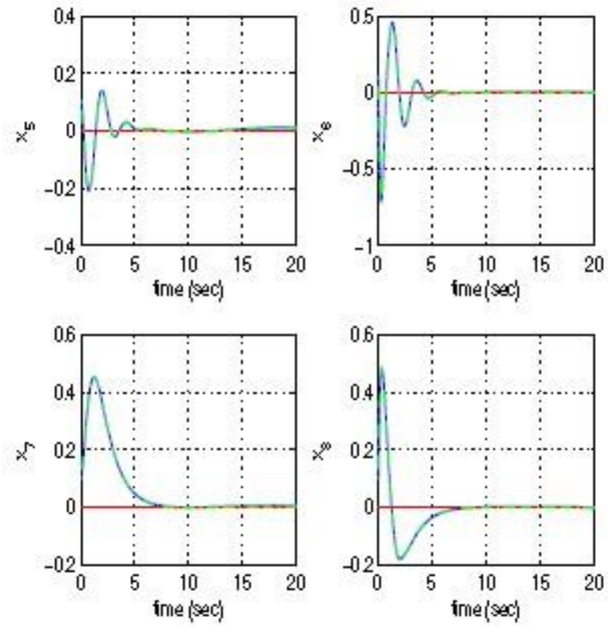
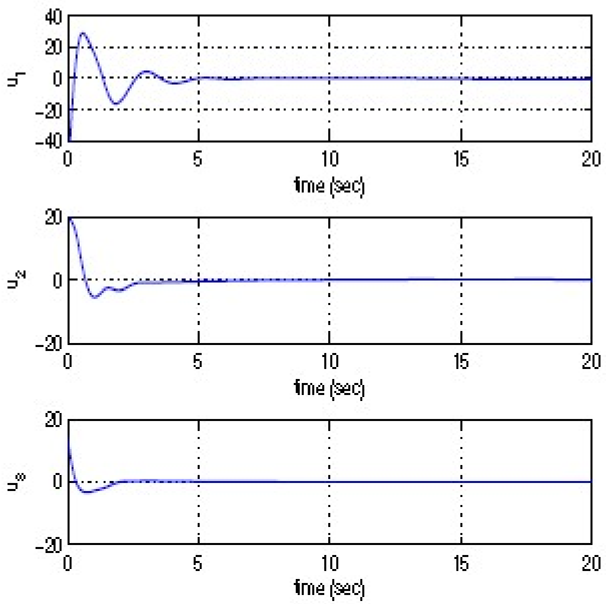


Fig 13(b) convergence of state variables x5 (rotation angle of the payload), x6 (rotational speed of the payload), x7 (roll angle of the UAV) and x8 (roll angular speed of the UAV) to their reference setpoints

6 . Example 2: Control and state estimation with approximate linearization

6.7. Simulation results

Setpoint 6



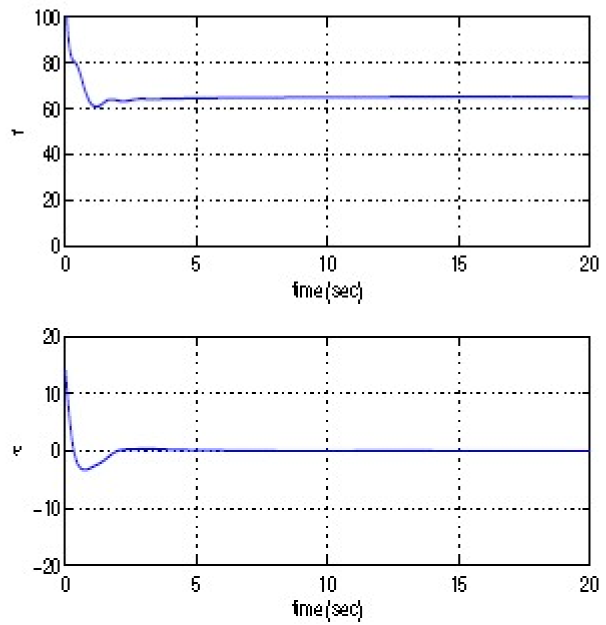


Fig. 14(a) control inputs to the UAV u_i i=1,2,3 computed through the solution of the nonlinear optimal control problem

Fig, 14(b) control inputs f (lift force of the UAV's motors) and (torque generated in aggregate by the motors of the UAV)

6 . Example 2: Control and state estimation with approximate linearization

6.8. Conclusions

● The use of UAVs in products transport and in ammunition tasks has necessitated the development of elaborated controllers for such robotic systems.

● In this article a novel nonlinear optimal controller has been applied to the dynamic model of a quadrotor UAV and suspended payload system.

● First, the dynamic model of the aerial robotic system has undergone approximate linearization around a temporary operating point (equilibrium) which was updated at each iteration of the control method.

● The linearization procedure relied on first- order Taylor series expansion of the state-space model of the robotic system and on the computation of the associated Jacobian matrices.

● For the approximately linearized model of the aerial robotic system an H-infinity feedback controller was designed.

● The global stability properties of the control scheme were proven through Lyapunov analysis.

● Finally, to implement state estimation-based control for the aerial robotic system, the H-infinity Kalman Filter has been used as a robust state estimator

7 . Example 3: Control and estimation with Lyapunov methods

7.1. Overview

- **Adaptive fuzzy control based** on **differential flatness theory** for **multivariable control** (dive-plane control) of **autonomous submarines**.

- It is proven that the **dynamic model of the submarine**, having as state variables the vessel's depth and its pitch angle, is a **differentially flat** one. This means that all its state variables and its control inputs can be written as differential functions of the flat output and its derivatives.

- By exploiting differential flatness properties the system's dynamic model is written in the **multivariable linear canonical (Brunovsky) form**, for which the design of a state feedback controller becomes possible.

- After this **transformation**, the **new control in**puts of the system contain **unknown nonlinear parts**, which are **identified with the use of neurofuzzy approximators**.

- The **learning procedure** for these estimators is determined by the requirement the **first derivative of the closed-loop's Lyapunov function to be a negative one**.

- Moreover, **the Lyapunov stability analysis** shows that **H-infinity tracking performance** is succeeded for the feedback control loop and this assures improved **robustness** to the aforementioned model uncertainty as well as to external perturbations.

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7 . Example 3: Control and estimation with Lyapunov methods

7.2. Dynamic model of the autonomous underwater vessel

The **multivariable model** of the **submarine’s dynamics** has as outputs

the **depth** of the submarine h
The **pitch angle** of the submarine θ

and as inputs

the **deflection angle of the hydroplanes** at the **front part** of vessel δB
the **deflection angle of the hydroplanes** located at the **rear** part of the vessel δS

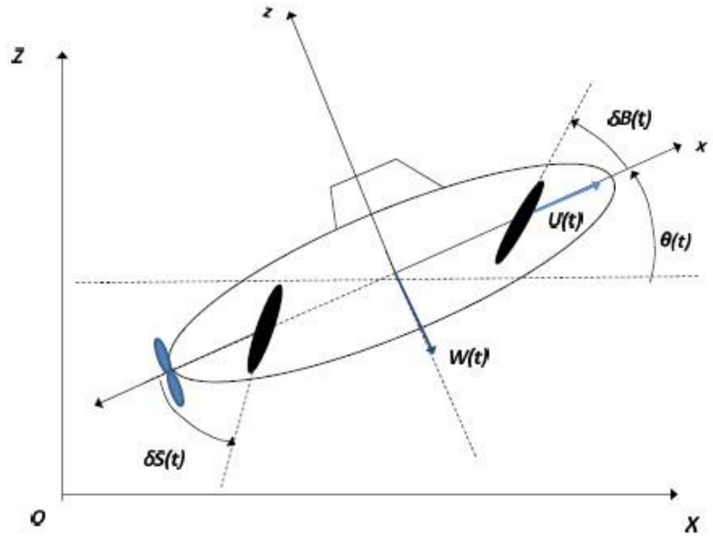


Fig.1 Diagram of the autonomous underwater vessel

7 . Example 3: Control and estimation with Lyapunov methods

7.2. Dynamic model of the autonomous underwater vessel

The **dvnamic model of the submarine** is written as:

$$\dot{w}(t) = \frac{Z'_w U}{L m'_3} w(t) + \frac{1}{m'_3}\dot{Z}'_{\dot\theta} + m')U\dot\theta(t) + \frac{Z'_{\dot Q} L}{m'_3}\dot{Q}(t) + \\ + \frac{Z'_{\delta B}U^2}{m'_3 L}\delta B(t) + \frac{Z'_{\delta S}U^2}{m'_3 L}\delta S(t) + \frac{Z_d(t)}{0.5\rho L^3 m'_3} + Z_\eta(w,q) \qquad (1)$$

$$\dot{Q}(t) = \frac{M'_{\dot w}}{L I'_2}\dot{w}(t) + \frac{M'_w U}{L^2 I'_2} w(t) + \frac{M'_{\dot\theta} U}{L I'_2}\dot\theta(t) + \\ + \frac{M'_{\delta B}U^2}{L^2 I'_2}\delta B(t) + \frac{M'_{\delta S}U^2}{L^2 I'_2}\delta S(t) + \frac{2mg(z_G - z_B)}{\rho L^5 I'_2}\theta(t) + \frac{M_d(t)}{0.5\rho L^5 I'_2} + M_\eta(w,q) \qquad (2)$$

w is the velocity along the z-axis, of the body-fixed frame

h is the depth of the vessel measured in the inertial coordinates system,

θ is the pitch angle

$Q = \dot\theta$ is the rate of change of the pitch angle

δB is the hydroplane deflection in the bow plane,

δS is the hydroplane deflection in the stern

Z_d, M_d are bounded disturbance inputs due to sea currents

$Z_\eta(w,q)$, $M_\eta(w,q)$ are disturbance inputs representing the vessel's cross-flow drag

$U = U_0$ denotes the x-axis (forward) velocity of the vessel.

7 . Example 3: Control and estimation with Lyapunov methods

7.2. Dynamic model of the autonomous underwater vessel

Indicative values of the parameters of the submarine.s dynamic model are:

Table I [1]

Parameters of the Submarine's dynamic model

Parameter Value	Parameter Value	Parameter Value
$Z'_w = -0.0110$	$Z'_{\dot{w}} = -0.0075$	$Z'_{\theta} = -0.0045$
$Z'_{\theta} = -0.0002$	$Z'_{\delta B} = -0.0025$	$Z'_{\delta S} = -0.0050$
$M'_w = 0.0030$	$M'_{\dot{w}} = -0.0002$	$M'_{\theta} = -0.0025$
$M'_{\dot{\theta}} = -0.0004$	$M'_{\delta B} = 0.0005$	$M'_{\delta S} = -0.0025$
$I'_y = 5.6867 \cdot 10e^{-4}$	$L = 286\text{ft}$	$m = 1.52 \cdot 10^5\,\text{slug}$
$Z_g - Z_B = -1.5\text{ft}$	$U = 8.43\text{ft/s}$	$\rho = 2.0\text{slug}/\text{ft}^3$
$I'_2 = I'_y - M'_B$	$m = 2m/(\rho L^3))$	$m'_3 = m' - Z'_w$

[1] K.Lee and S,Singh,Journal of Systems and Control Engineering, vol. 328, no. 3, 2014

These can be obtained directly from the design characteristics of the vessel or indirectly through an **identification procedure** in the sense of nonlinear least squares or nonlinear Kalman Filtering

However, since **adaptive control is a model-free control method**, there is **no need about prior knowledge** of these parameters' values..

Adaptive control assures stability of the control loop **under** unknown dynamic model parameters and unknown **external perturbations and disturbances** ..

7 . Example 3: Control and estimation with Lyapunov methods

7.2. Dynamic model of the autonomous underwater vessel

The **dynamic model of the submarine** can be written in **matrix form**:

$$\begin{pmatrix} \dot{w} \\ \dot{Q} \end{pmatrix} = \begin{pmatrix} f_W(w,\theta,Q,t) \\ f_\theta(w,\theta,Q,t) \end{pmatrix} + B_o u \qquad (3)$$

where the **control input vector** is $u = [\delta B(t) \ \delta S(t)]^T$

and is generated by **electric actuators** that rotate the hydroplanes. Therefore the control input describes actually **voltage or current signals** that define the turn angle of the rotor of these electric actuators.

This indicates clearly the significance of **electric actuators** in the **submarine's propulsion**.

In this description:

$$\begin{pmatrix} f_w(w,\theta,Q,t) \\ f_\theta(f_W(w,\theta,Q,t) \end{pmatrix} = M^{-1} \begin{pmatrix} \frac{Z_w' U}{L m_3'} w(t) + \frac{1}{m_3'} \dot{Z}_\theta' + m') U \dot{\theta}(t) + \frac{Z_{\dot{Q}}' L}{m_s'} \dot{Q}(t) + \frac{Z_d(t)}{0.5\rho L^3 m_3'} + Z_\eta(w,q) \\ \frac{M_{\dot{w}}'}{L I_o'} \dot{w}(t) + \frac{M_v' U}{L^2 I_o'} w(t) + \frac{M_{\dot{\theta}}' U}{L I_o'} \dot{\theta}(t) + \frac{2mg(z_G - z_B)}{\rho L^5 I_o'} \theta(t) + \frac{M_d(t)}{0.5\rho L^5 I_o'} + M_\eta(w,q) \end{pmatrix}$$

while for matrices M and B_o it holds

$$M = \begin{pmatrix} 1 & -Z_{\dot{Q}} L / m_3' \\ -M_{\dot{w}} (L {I_2'}^{-1}) & 1 \end{pmatrix} \quad B_o = \begin{pmatrix} \frac{Z_{\delta B}' U^2}{m_3' L} & \frac{Z_{\delta S}' U^2}{m_3' L} \\ \frac{M' \delta B U^2}{L^2 I_2'} & \frac{M' \delta S U^2}{L^2 I_2'} \end{pmatrix}$$

7 . Example 3: Control and estimation with Lyapunov methods

7.2. Dynamic model of the autonomous underwater vessel

It holds that the **depth of the vessel** measured **in the inertial reference frame** and the velocity **w** of the submarine along the z-axis of the **body-fixed frame** are related as follows:

$$\begin{array}{c}\dot{h} = w cos(\theta) - U_o sin(\theta) \Rightarrow \\ \ddot{h} = \dot{w} cos(\theta) - w sin(\theta)\dot{\theta} - U_o cos(\theta)\dot{\theta} \Rightarrow \\ \ddot{h} = \dot{w} cos(\theta) - w Q sin(\theta) - U_o Q cos(\theta)\end{array} \quad (4)$$

From the above relation one can compute about the **diving speed** of the vessel:

$$w = (cos(\theta)^{-1})(\dot{h} + U_o sin(\theta)) \quad (5)$$

Moreover, from Eq (3) one has:

$$\begin{array}{c}\dot{w} = f_w(w, \theta, Q, t) + B_{o_{11}} u_1 + B_{o_{12}} u_2 \\ \dot{Q} = f_\theta(w, \theta, Q, t) + B_{o_{21}} u_1 + B_{o_{22}} u_2\end{array} \quad (6)$$

Substituting Eq. (5) and the first row of Eq. (6) into Eq. (4) one gets

$$\ddot{h} = [f_w(w, \theta, Q, t) + B_{o_{11}} u_1 + B_{o_{12}} u_2] cos(\theta) - \frac{(\dot{h} + U_0 sin(\theta))}{cos(\theta)} Q sin(\theta) - U_0 Q cos(\theta) \quad (7)$$

7 . Example 3: Control and estimation with Lyapunov methods

7.2. Dynamic model of the autonomous underwater vessel

Next, by denoting:

$$f_w(w,\theta,Q,t)=g_h(h,\dot{h},\theta,\dot{\theta},t)$$
$$f_\theta(w,\theta,Q,t)=g_\theta(h,\dot{h},\theta,\dot{\theta},t)$$

And by substituting this relation in Eq. 7, together with $Q=\dot{\theta}$ one obtains:

$$\ddot{h}=g_h(h,\dot{h},\theta,\dot{\theta},t)cos(\theta)-\frac{(\dot{h}+U_0 sin(\theta))}{cos(\theta)}\dot{\theta}sin(\theta)-U_0\dot{\theta}cos(\theta)+$$
$$+B_{0_{11}}cos(\theta)u_1+B_{0_{12}}cos(\theta)u_2$$

$$\ddot{\theta}=g_\theta(h,\dot{h},\theta,\dot{\theta},t)+B_{0_{21}}u_1+B_{0_{22}}u_2$$

Then, by defining the **state vector** $x=[h,\dot{h},\theta,\dot{\theta}]^T$

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_3\end{pmatrix}=\begin{pmatrix}g_b(x,t)cos(x_3)-\frac{x_4+U_0 sin(x_3)}{cos(x_3)}x_4 sin(x_3)-U_0x_4cos(x_3)\\ g_\theta(x,t)\end{pmatrix}+\begin{pmatrix}B_{0_{11}} & B_{0_{12}}\\ B_{0_{21}} & B_{0_{22}}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} \quad (8)$$

From Eq. 8 one finally arrives at the **MIMO state-space description** of the submarine

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_3\end{pmatrix}=\begin{pmatrix}f_1(x,t)\\ f_2(x,t)\end{pmatrix}+\begin{pmatrix}g_{11}(x,t) & g_{12}(x,t)\\ g_{21}(x,t) & g_{22}(x,t)\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} \quad (9)$$

7 . Example 3: Control and estimation with Lyapunov methods

7.3. Differential flatness properties of the autonomous underwater vessel

- **Differential flatness theory** has been developed as a **global linearization control method** by M. Fliess (Ecole Polytechnique, France) and co-researchers (Lévine, Rouchon, Mounier, Rudolph, Petit, Martin, Zhu, Sira-Ramirez et. al)

- A dynamical system can be written in the ODE form $S_i(w,\dot{w},\ddot{w},\ldots,w^{(i)}),\ \ i=1,2,\ldots,q$

  where $w^{(i)}$ stands for the i-th derivative of either a state vector element or of a control input

- The system is said to be **differentially flat** with respect to the **flat output**

$$y_i=\phi(w,\dot{w},\ddot{w},\ldots,w^{(a)}),\ i=1,\ldots,m \quad \text{where} \quad y=(y_1,y_2,\ldots,y_m)$$

if the following two conditions are satisfied

(i) There does not exist any differential relation of the form

$$R(y,\dot{y},\ddot{y},\ldots,y^{(\beta)})=0$$

which means that **the flat output and its derivatives are linearly independent**

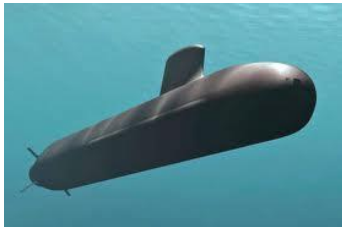
(ii) All system variables are **functions of the flat output and its derivatives**

$$w^{(i)}=\psi(y,\dot{y},\ddot{y},\ldots,y^{(\gamma_i)})$$

7 . Example 3: Control and estimation with Lyapunov methods

7.3. Differential flatness properties of the autonomous underwater vessel

The proposed adaptive control method is based on the **transformation** of the vessel's model into the **linear canonical form**, and this transformation is succeeded by exploiting the system's differential flatness properties



• **All single input vessel models** are differentially flat and can be transformed into the linear canonical form

One has to define also which are the **MIMO vessel models** which are differentially flat.

• Differential flatness holds for **MIMO vessel models** that admit **static feedback linearization** and which can be transformed into the linear canonical form through a change of variables (diffeomorphism) and feedback of the state vector. This **is the case of the submarine's model**

• Differential flatness holds for **MIMO vessel models** that admit **dynamic feedback linearization**, This **is the case of underactuated vessel models** (e.g. hovercraft) In the latter case the state vector of the system is extended by considering as additional flat outputs some of the control inputs and their derivatives

• Finally, a more rare case is the so-called **Liouvillian systems.** These are systems for which differential flatness properties hold for part of their state vector (constituting a flat subsystem) while the non-flat state variables can be obtained by integration of the elements of the flat subsystem.

7 . Example 3: Control and estimation with Lyapunov methods

7.3. Differential flatness properties of the autonomous underwater vessel

Next, by denoting the **flat output of the submarine** as:

$$y = [x_1, x_3]^T = [h, \theta]^T$$

it can be proven that the submarine's dynamic model is a **differentially flat** one

This means that all its **state variables** and its **control inputs** can be expressed as **differential functions of the flat output**

From Eq. 9 one gets $x_2 = \dot{x}_1$ and $x_4 = \dot{x}_3,$ which means

$$x_2 = [1\ 0]\dot{y} \quad x_4 = [0\ 1]\dot{y} \qquad (10)$$

Again, from Eq. 9 one gets

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \end{pmatrix}^{-1} \left(\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} - \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \right)$$

which means

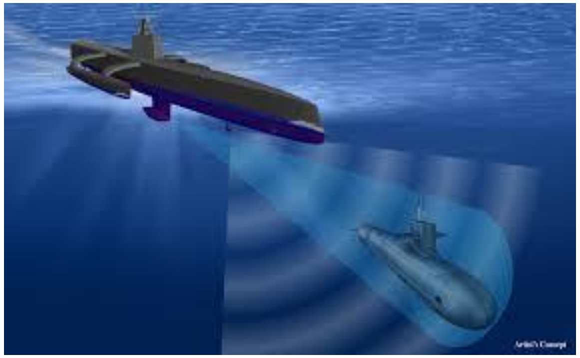
$$u_1 = f_a(y, \dot{y}, \ddot{y}) \quad u_2 = f_b(y, \dot{y}, \ddot{y}). \qquad (11)$$

Eq. 10 and Eq. 11 confirm that the submarine's model is a differentially flat one.

7 . Example 3: Control and estimation with Lyapunov methods

7.3. Differential flatness properties of the autonomous underwater vessel

The differential flatness property of the submarine's model is important because it means that the vessel's model can be transformed into the **MIMO linear canonical (Brunovsky) form** through a change of its state variables (diffeomorphism)



By defining the **new state variables of the vessel**

$$y_1 = x_1, \quad y_2 = \dot{y}_1, \quad y_3 = x_2, \quad y_4 = \dot{y}_3$$

and by defining **the transformed control inputs** of the vessel

$$\begin{aligned} v_1 &= f_1(x,t) + g_{11}u_1 + g_{12}u_2 \\ v_2 &= f_2(x,t) + g_{21}u_1 + g_{22}u_2 \end{aligned} \qquad (12)$$

one obtains the **linearized and decoupled state-space model** of the submarine

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad (13)$$

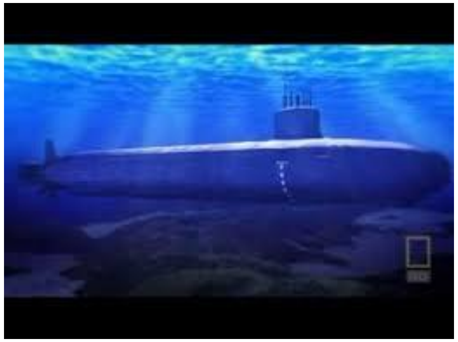
for which the design of a **state-feedback controller** is possible

7 . Example 3: Control and estimation with Lyapunov methods

7.4. Design of a state-feedback controller for the autonomous underwater vessel

For the **transformed state-space model** of the vessel

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (13)$$



It is considered that the complete state vector is measurable

$$y = [h, \dot{h}, \theta, \dot{\theta}]$$

Then, to succeed tracking of the **reference setpoint**

$$y^d = [y_1^d, y_2^d, y_3^d, y_4^d]^T = [x_1^d, \dot{x}_1^d, x_2^d, \dot{x}_2^d]^T$$

the **feedback control inputs** should be chosen as

$$\begin{aligned} v_1 &= \ddot{y}_1^d - k_d^1(\dot{y}_1 - \dot{y}_1^d) - k_p^1(y_1 - y_1^d) \\ v_2 &= \ddot{y}_3^d - k_d^2(\dot{y}_3 - \dot{y}_3^d) - k_p^2(y_3 - y_3^d) \end{aligned} \quad (14)$$

7 . Example 3: Control and estimation with Lyapunov methods

7.4. Design of a state-feedback controller for the autonomous underwater vessel

By substituting Eq. (14) Into Eq. (13) one obtains the

tracking error dynamics for the submarine

$$\ddot{e}_1 + k_d^1 \dot{e}_1 + k_p^1 e_1 = 0 \qquad \ddot{e}_2 + k_d^2 \dot{e}_2 + k_p^2 e_2 = 0 \qquad (15)$$

where the tracking error is defined as $e_1 = y_1 - y_1^d, \quad e_2 = y_3 - y_3^d$

By selecting the feedback control gains $k_p^i, k_d^i \quad i=1,2$ so as the **characteristic polynomials**

$$p_1(s) = s^2 + k_d^1 s + k_p^1 \qquad p_2(s) = s^2 + k_d^2 s + k_p^2 \qquad (16)$$

to have roots explicitly in the **left complex semiplane**, it is assured that

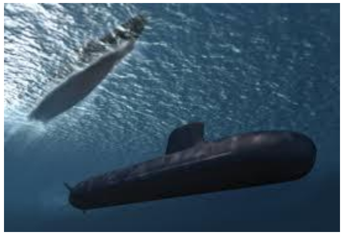
$$\lim_{t \to \infty} e_i(t) = 0 \quad i=1,2$$

Finally, the feedback **control input** that is **actually exerted on the submarine** is .

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} g_{11}(x,t) & g_{12}(x,t) \\ g_{21}(x,t) & g_{22}(x,t) \end{pmatrix}^{-1} [\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}] \qquad (17)$$

7 . Example 3: Control and estimation with Lyapunov methods

7.5. Design of an adaptive controller for the autonomous underwater vessel



For the **differentially flat MIMO model of the submarine** one has the dynamics

$$\ddot{x}_1 = f_1(x,t) + g_1(x,t)u + \tilde{d}_1$$

$$\ddot{x}_3 = f_2(x,t) + g_2(x,t)u + \tilde{d}_2$$

The following **control input is** considered

$$u = \begin{bmatrix} \hat{g}_1(x,t) \\ \hat{g}_2(x,t) \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \ddot{x}_1^d \\ \ddot{x}_3^d \end{bmatrix} - \begin{bmatrix} \hat{f}_1(x,t) \\ \hat{f}_2(x,t) \end{bmatrix} - \begin{bmatrix} K_1^T \\ K_2^T \end{bmatrix} e + \begin{bmatrix} u_{c_1} \\ u_{c_2} \end{bmatrix} \right\}$$

where $\hat{f}$ and $\hat{g}$ stand for estimates of the unknown nonlinear terms f and g

These estimates are provided by neurofuzzy approximators or other nonlinear regressors

This results in **tracking error dynamics** of the form

$$\dot{e} = (A - BK^T)e + Bu_c + B\left\{ \begin{bmatrix} f_1(x,t) - \hat{f}_1(x,t) \\ f_2(x,t) - \hat{f}_2(x,t) \end{bmatrix} + \begin{bmatrix} g_1(x,t) - \hat{g}_1(x,t) \\ g_2(x,t) - \hat{g}_2(x,t) \end{bmatrix} \begin{bmatrix} \hat{g}_1(x,t) \\ \hat{g}_2(x,t) \end{bmatrix}^{-1} u + \tilde{d} \right\}$$

where matrices A,B,K are defined as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, K^T = \begin{bmatrix} K_1^1 & K_2^1 & K_3^1 & K_4^1 \\ K_1^2 & K_2^2 & K_3^2 & K_4^2 \end{bmatrix}$$

7 . Example 3: Control and estimation with Lyapunov methods

7.5. Design of an adaptive controller for the autonomous underwater vessel

The **nonlinear regressors** (neurofuzzy approximators) **consist** of the **kernel functions** and **weights functions.** Unlike SISO systems, in the case of MIMO dynamics the kernel and weights functions are not represented as vectors but **take the form of matrices**. Thus one has:

$$\hat{f}(x|\theta_f)=\Phi_f(x)\theta_f \quad \text{and} \quad \hat{g}(x|\theta_g)=\Phi_g(x)\theta_g$$

Kernel and weights functions for the **approximation of the nonlinear dynamics *f*:**

$$\Phi_f(x)=\begin{bmatrix} \phi_f^{1,1}(x) & \phi_f^{1,2}(x) & \dots & \phi_f^{1,N}(x) \\ \phi_f^{2,1}(x) & \phi_f^{2,2}(x) & \dots & \phi_f^{2,N}(x) \\ \dots & \dots & \dots & \dots \\ \phi_f^{n,1}(x) & \phi_f^{n,2}(x) & \dots & \phi_f^{n,N}(x) \end{bmatrix} \qquad \theta_f^T=\begin{bmatrix}\theta_f^1 & \theta_f^2 & \dots & \theta_f^N\end{bmatrix}$$

Kernel and weights functions for the **approximation of the nonlinear dynamics *g***:

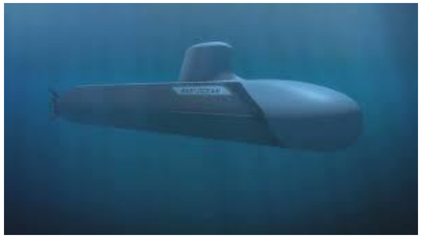
$$\Phi_g(x)=\begin{bmatrix} \phi_g^{1,1}(x) & \phi_g^{1,2}(x) & \dots & \phi_g^{1,N}(x) \\ \phi_g^{2,1}(x) & \phi_g^{2,2}(x) & \dots & \phi_g^{2,N}(x) \\ \dots & \dots & \dots & \dots \\ \phi_g^{n,1}(x) & \phi_g^{n,2}(x) & \dots & \phi_g^{n,N}(x) \end{bmatrix} \qquad \theta_g=\begin{bmatrix} \theta_{g1}^1 & \theta_{g1}^2 & \dots & \theta_{g1}^p \\ \theta_{g2}^1 & \theta_{g2}^2 & \dots & \theta_{g2}^p \\ \dots & \dots & \dots & \dots \\ \theta_{gN}^1 & \theta_{gN}^2 & \dots & \theta_{gN}^p \end{bmatrix}$$

7 . Example 3: Control and estimation with Lyapunov methods

7.5. Design of an adaptive controller for the autonomous underwater vessel

The **weight functions** of the neurofuzzy approximators **are learned** through an **adaptation procedure** that is determined by **Lyapunov stability analysis** for the submarine's model.

The following quadratic **Lyapunov function** is defined:



$$V=\frac{1}{2}e^TPe+\frac{1}{2\gamma_1}\tilde{\theta}_f^T\tilde{\theta}_f+\frac{1}{2\gamma_2}tr[\tilde{\theta}_g^T\tilde{\theta}_g]$$

e: state vector tracking error

$\tilde{\theta}_f=\theta_f-\theta_f^*$: Difference of the weights from the value that succeeds exact estimation of *f*

$\tilde{\theta}_g=\theta_g-\theta_g^*$: Difference of the weights from the value that succeeds exact estimation of *g*

Differentiating one obtains: $\dot{V}=\frac{1}{2}\dot{e}^TPe+\frac{1}{2}e^TP\dot{e}+\frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T\tilde{\theta}_f+\frac{1}{\gamma_2}tr[\dot{\tilde{\theta}}_g^T\tilde{\theta}_g]$

The associated tracking error dynamics is:

$$\dot{e}=(A-BK^T)e+Bu_c+B\{\begin{bmatrix}f_1(x,t)-\hat{f}_1(x,t)\\ f_2(x,t)-\hat{f}_2(x,t)\end{bmatrix}+\begin{bmatrix}g_1(x,t)-\hat{g}_1(x,t)\\ g_2(x,t)-\hat{g}_2(x,t)\end{bmatrix}\begin{bmatrix}\hat{g}_1(x,t)\\ \hat{g}_2(x,t)\end{bmatrix}^{-1}u+\tilde{d}\}$$

The **effect of modelling errors** is denoted by:

$$w=\begin{bmatrix}f_1(x,t)-\hat{f}_1(x,t)\\ f_2(x,t)-\hat{f}_2(x,t)\end{bmatrix}+\begin{bmatrix}g_1(x,t)-\hat{g}_1(x,t)\\ g_2(x,t)-\hat{g}_2(x,t)\end{bmatrix}\begin{bmatrix}\hat{g}_1(x,t)\\ \hat{g}_2(x,t)\end{bmatrix}^{-1}u$$

7 . Example 3: Control and estimation with Lyapunov methods

7.5. Design of an adaptive controller for the autonomous underwater vessel

Thus one obtains the following **tracking error dynamics**:

$$\dot{e} = (A - BK^T)e + Bu_c + B(w + \tilde{d})$$

The **first derivative of the Lyapunov function** becomes:

$$\dot{V} = \frac{1}{2}\{e^T (A - BK^T)^T + u_c^T B^T + (w + \tilde{d})^T B^T\}Pe + \frac{1}{2}e^T P\{(A - BK^T)e + Bu_c + B(w + \tilde{d})\}$$

$$+\frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T \tilde{\theta}_f + \frac{1}{\gamma_2} tr[\dot{\tilde{\theta}}_g^T \tilde{\theta}_g]$$

and after intermediate terms substitution one obtains:

$$\dot{V} = \frac{1}{2}e^T\{(A - BK^T)^T P + P(A - BK^T)\}e + \frac{1}{2}2e^T PBu_c + \frac{1}{2}2B^T Pe(w + \tilde{d})$$

$$+\frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T \tilde{\theta}_f + \frac{1}{\gamma_2} tr[\dot{\tilde{\theta}}_g^T \tilde{\theta}_g]$$

Assumption 1: the positive definite and symmetric matrix P is chosen as solution of the Riccati equation:

$$(A - BK^T)^T P + P(A - BK^T) - PB(\frac{2}{r} - \frac{1}{\rho^2})B^T P + Q = 0 \quad (18)$$

7 . Example 3: Control and estimation with Lyapunov methods

7.5. Design of an adaptive controller for the autonomous underwater vessel

Using as supervisory control input $u_c = -\frac{1}{r}B^T Pe$ one obtains:

$$\dot{V} = \frac{1}{2}e^T\{-Q + PB(\frac{2}{r} - \frac{1}{\rho^2})B^T P\}e + e^T PB\{-\frac{1}{r}B^T Pe\} + B^T P(w + \tilde{d}) +$$

$$+\frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T \tilde{\theta}_f + \frac{1}{\gamma_2}tr[\dot{\tilde{\theta}}_g^T \tilde{\theta}_g]$$

which can be written in the form:

$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) + \frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T \tilde{\theta}_f + \frac{1}{\gamma_2}tr[\dot{\tilde{\theta}}_g^T \tilde{\theta}_g]$$

Next, substituting: $\dot{\tilde{\theta}}_f = \dot{\theta}_f - \dot{\theta}_f^* = \dot{\theta}_f$ and $\dot{\tilde{\theta}}_g = \dot{\theta}_g - \dot{\theta}_g^* = \dot{\theta}_g$

i.e: $\dot{\theta}_f = -\gamma_1 \Phi(x)^T B^T Pe$ and $\dot{\theta}_g = -\gamma_2 \Phi(x)^T B^T Peu^T$

the following form of the **derivative of the Lyapunov function** is obtained:

$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) +$$

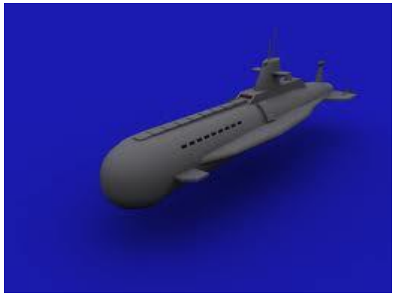
$$+\frac{1}{\gamma_1}(-\gamma_1)e^T PB\Phi(x)(\theta_f - \theta_f^*) + \frac{1}{\gamma_2}(-\gamma_2)tr[ue^T PB\Phi(x)(\theta_g - \theta_g^*)]$$

7 . Example 3: Control and estimation with Lyapunov methods

7.5. Design of an adaptive controller for the autonomous underwater vessel

Taking into account that $u \in R^{2\times1}$ and $e^T PB(\hat{g}(x|\theta_g) - \hat{g}(x|\theta_g^*)) \in R^{1\times2}$

the following form is obtained for the **Lyapunov function derivative** :



$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) +$$

$$+\frac{1}{\gamma_1}(-\gamma_1)e^T PB\Phi(x)(\theta_f - \theta_f^*) + \frac{1}{\gamma_2}(-\gamma_2)tr[e^T PB(\hat{g}(x|\theta_g) - \hat{g}(x|\theta_g^*))u]$$

and since $e^T PB(\hat{g}(x|\theta_g) - \hat{g}(x|\theta_g^*))u \in R^{1\times1}$ it holds that

$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) +$$

$$+\frac{1}{\gamma_1}(-\gamma_1)e^T PB\Phi(x)(\theta_f - \theta_f^*) + \frac{1}{\gamma_2}(-\gamma_2)e^T PB(\hat{g}(x|\theta_g) - \hat{g}(x|\theta_g^*))u$$

Using the following description for the model approximation error:

$$w_a = [\hat{f}(x|\theta_f^*) - \hat{f}(x|\theta_f)] + [\hat{g}(x|\theta_f^*) - \hat{g}(x|\theta_f)]u$$

the equation of the Lyapunov function derivative becomes:

$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) + e^T PBw_a$$

7 . Example 3: Control and estimation with Lyapunov methods

7.5. Design of an adaptive controller for the autonomous underwater vessel

and denoting the disturbances and modelling error terms as: $w_1 = w + \tilde{d} + w_a$

one has: $$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T e + e^T PBw_1$$

or: $$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T e + \frac{1}{2}e^T PBw_1 + \frac{1}{2}w_1^T B^T Pe$$

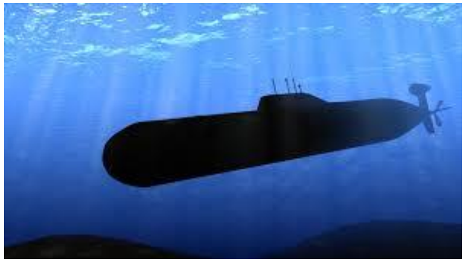
Next **the following inequality** is used:

Lemma: It holds that $$\frac{1}{2}e^T Pw_1 + \frac{1}{2}w_1^T B^T Pe - \frac{1}{2\rho^2}e^T PBB^T Pe \leq \frac{1}{2}\rho^2 w_1^T w_1 \quad (19)$$

Proof:

The binomial $(\rho a - \frac{1}{\rho}b)^2 \geq 0$ is considered. Expanding the left part of the above inequality one gets

$$\rho^2 a^2 + \frac{1}{\rho^2}b^2 - 2ab \geq 0 \Rightarrow \frac{1}{2}\rho^2 a^2 + \frac{1}{2\rho^2}b^2 - ab \geq 0 \Rightarrow$$
$$ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2 a^2 \Rightarrow \frac{1}{2}ab + \frac{1}{2}ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2 a^2$$



By substituting $a = w_1$ and $b = \bar{e}^T P_2 B$ one gets

$$\frac{1}{2}w_1^T B^T P_2 \bar{e} + \frac{1}{2}\bar{e}^T P_2 B w_1 - \frac{1}{2\rho^2}\bar{e}^T P_2 B B^T P_2 \bar{e} \leq \frac{1}{2}\rho^2 w_1^T w_1$$

7 . Example 3: Control and estimation with Lyapunov methods

7.5. Design of an adaptive controller for the autonomous underwater vessel

By substituting Eq. 19 into the relation of the derivative of the Lyapunov function gives:

$$\dot{V} \leq -\frac{1}{2}e^T Q e + \frac{1}{2}\rho^2 w_1^T w_1 \qquad (20)$$

This is the **H-infinity tracking performance criterion** which means that for bounded disturbance and modelling error the control law results in very small bounded tracking error:

It is noted that, by choosing the **attenuation coefficient ρ** to be sufficiently small, the right part of Eq. 20 can be always made to be upper bounded by zero.

In such a case the **asymptotic stability condition** is clear to hold..

The minimum value of ρ for which a solution of the Riccati Eq. 18 exists, is the one that provides the control loop with maximum robustness.

Moreover, if $\int_0^\infty \| w_1 \|^2 dt \leq M_w$ one has the following integral:

$$\int_0^T \dot{V}(t)dt \leq -\frac{1}{2}\int_0^T \| e(t) \|^2 dt + \frac{1}{2}\rho^2 \int_0^T \| w_1 \|^2 \, dt \Rightarrow 2V(T) + \int_0^T \| e(t) \|_Q^2 \, dt \leq 2V(0) + \rho^2 \int_0^T \| w_1 \|^2 \, dt$$

which means that: $\int_0^\infty \| e \|_Q^2 \, dt \leq 2V(0) + \rho^2 M_w$ and from **Barbalat's Lemma** one has that

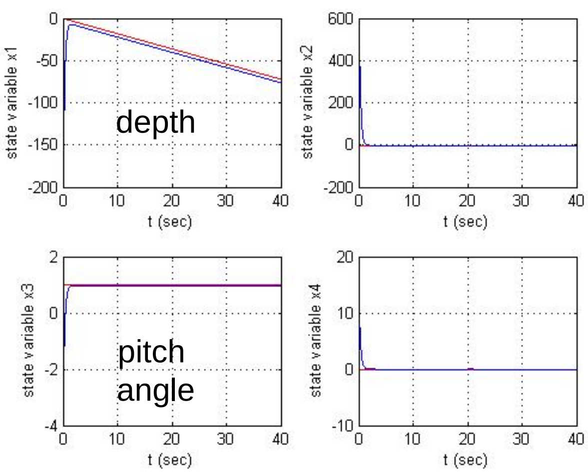
$\lim_{t \to \infty} e(t) = 0$ which confirms again that the tracking error vanishes

7 . Example 3: Control and estimation with Lyapunov methods

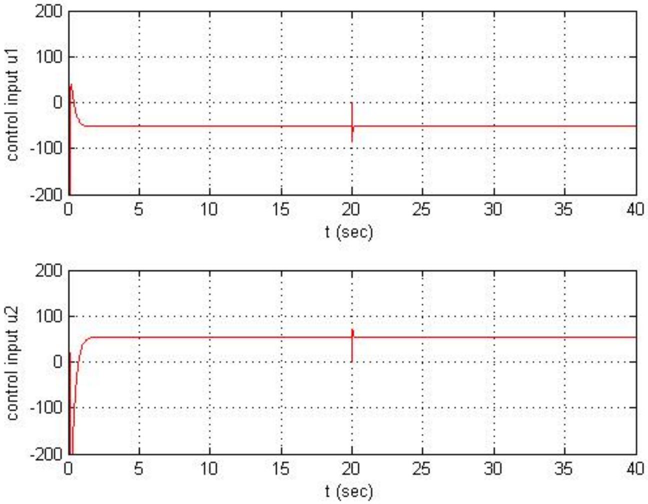
7.6. Simulation results

• In the simulation tests, the **dynamic model of the submarine** was considered to be **completely unknown** and was identified in real-time by the previously analyzed nonlinear regressors

• The **estimated unknown dynamics** of the system **was used in the computation of the control inputs** (generated by the electric actuators of the hydroplanes) which were finally exerted on the submarine’s model.



Setpoint 1:



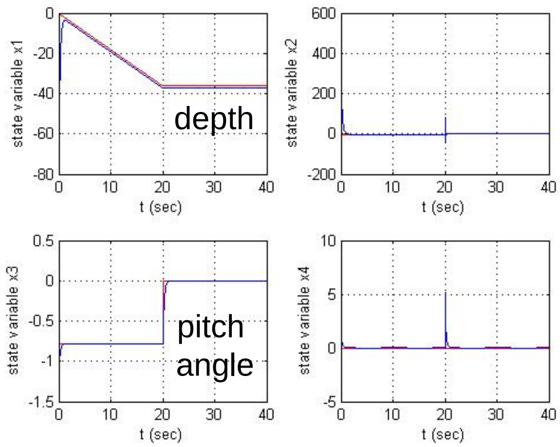
state variables $x_i, \ i = 1, \cdots, 4$

Variations of the control inputs

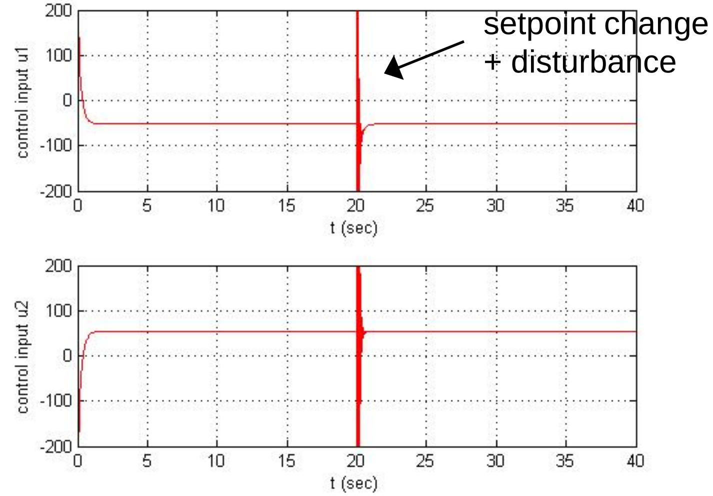
7 . Example 3: Control and estimation with Lyapunov methods

7.6. Simulation results

Setpoint 2:

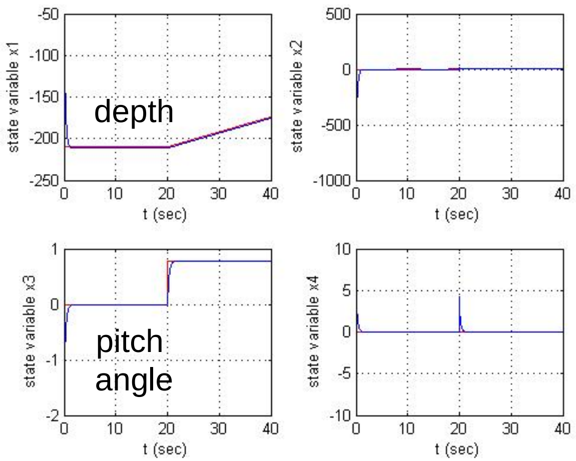


state variables x_i, $i = 1, \cdots, 4$

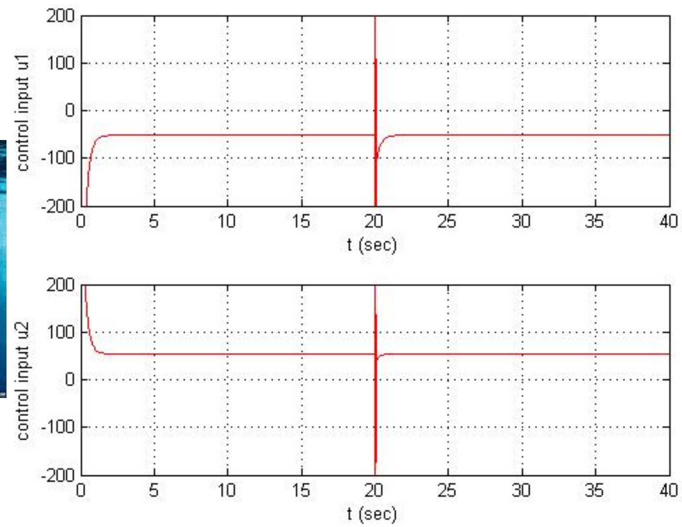


Variations of the control inputs

Setpoint 3:



state variables x_i, $i = 1, \cdots, 4$



Variations of the control inputs

7 . Example 3: Control and estimation with Lyapunov methods

7.7. Conclusions

• By exploiting the **differential flatness** properties of the **MIMO nonlinear model of the submarine** the system was transformed into the **linear canonical (Brunovsky) form.** For the latter description the design of a feedback controller was possible.

• Moreover, to cope with **unknown nonlinear terms** appearing in the new control inputs of the transformed state-space description of the submarine, the use of nonlinear regressors (neurofuzzy approximators) has been proposed..

• These estimators were online trained to identify the unknown dynamics and the learning procedure was determined by the requirement the derivative of the Lyapunov function to be negative

• Through **Lyapunov stability analysis** it was proven that the closed loop satisfies the **H-infinity tracking performance criterion**, and this assures improved robustness against model uncertainties and external perturbations.

• The computation of the control input required the **solution of an algebraic Riccati equation**. Suitable selection of the attenuation coefficient ρ in this equation assures asymptotic stability and provides maximum robustness.

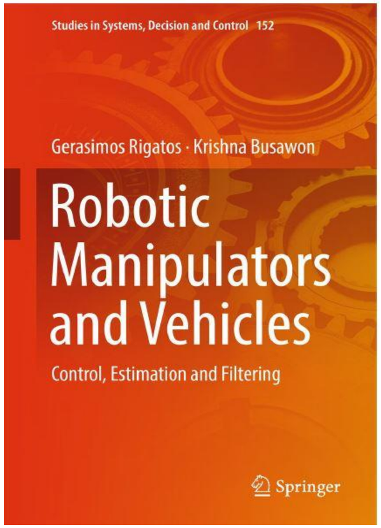
• The proposed flatness-based adaptive fuzzy control method is generic and **can be applied to a wide class of vessels**, such as surface vessels or AUVs and submersibles,inIcluding also the case of underactuated vessels.

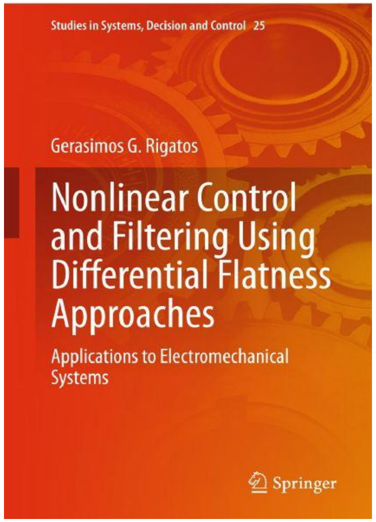
8. Final conclusions

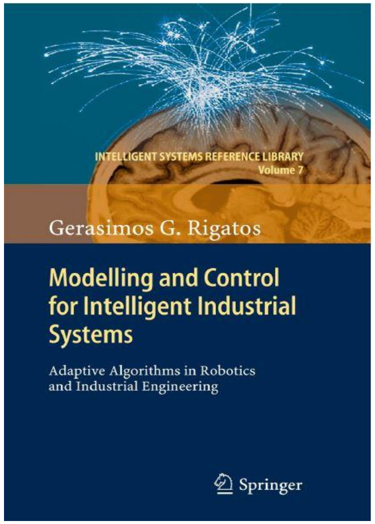
● Methods for nonlinear control and state estimation in autonomous robotic vehicles have been developed

● The main approaches for nonlinear control have been: (i) **control with global linearization** method (ii) **control with approximate (asymptotic) linearization** methods (iii) **control with Lyapunov theory methods (adaptive control)** in case that the robotic or kinematic model of the vehicle is unknown

● The main approaches for nonlinear state estimation are: (i) nonlinear state estimation with methods of global linearization (ii) nonlinear state estimation with methods of approximate (asymptotic) linearization







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