Differential flatness properties and control of distributed parameter systems in finance

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1. Outline

• **Pricing of commodities** (e.g. oil, carbon, mining products, electric power, agricultural crops, etc.) is **vital for the majority of transactions** taking place in financial markets. A method for **feedback control of commodities pricing dynamics** is developed.

• The **PDE model of the commodities price dynamics** is shown to be **equivalent to a multi-asset Black-Scholes PDE**. Actually it is a diffusion process evolving in a 2D assets space, where **the first asset is the commodity's spot price** and **the second asset is the convenience yield**.

• By applying **semi-discretization** and a **finite differences scheme** this **multi-asset PDE** is transformed into a **state-space model** consisting of **nonlinear differential equations**.

• The **controller design** proceeds by showing that the **state-space model of the commodities PDE** stands for a **differentially flat system**. Next, **for each subsystem which is related to a nonlinear ODE**, a **virtual control input is computed**, that can invert the subsystem's dynamics and can eliminate the tracking error.

• From the **last row of the state-space description**, the **control input (boundary condition)** that is actually applied to the multi-factor commodities' PDE system is found.

• By showing the **feasibility of such a control method** it is also proven that **through selected purchase and sales** during the trading procedure **the price of the negotiated commodities** can be made to converge and **stabilize at specific reference values**.

2. The Commodities Price PDE

• **Advanced pricing models for commodities** are **not only based on the spot pricing** approach but **reflect the dynamics of prices within long-term contracts**

• This **dynamics** can be expressed either **in the form of stochastic differential equations** or equivalently in the form of **partial differential equations (PDEs)**

• A **pricing approach in long-term contracts** is based on the **use of the commodities price PDE**

• In the **two-factor model the distribution** of the commodity's price $F(S,\delta,t)$ is now dependent on **two variables**, where the first one is the **sport price of the commodity** S and the second is the so-called **convenience yield** δ or long-term price.

Now the variation of S and δ is described by the **stochastic processes**

$$dS = (\mu-\delta)Sdt + \sigma_1 Sdz_1 \qquad (1)$$

$$d\delta = \kappa(a-\delta)dt + \sigma_2 Sdz_2 \qquad (2)$$

where the increments to standard Brownian motion which are correlated with

$$dz_1 dz_2 = \rho dt$$

. Defining again $X = lnS$ and applying Ito's Lemma the process for the log price becomes

$$dX = (\mu-\delta-\tfrac{1}{2}\sigma_1^2)dt + \sigma_1 dz_1 \qquad (3)$$

2. The Commodities Price PDE

The **stochastic processes** for the underlying factors, that is the *spot price* and the **convenience yield** can be also written as

$$dS = (r-\delta)Sdt + \sigma_1 S dz_1^* \quad (4)$$

$$d\delta = [\kappa(a-\delta) - \lambda]dt + \sigma_2 dz_2^* \quad (5)$$

$$dz_1^* dz_2^* = \rho dt \quad (6)$$

where λ is the market price of the convenience yield risk

Futures prices then can be equivalently computed from the solution of the following partial differential equation, which stands for the **2-factor PDE model of the commodities price**

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \sigma_1\sigma_2\rho S\frac{\partial^2 F}{\partial S\partial\delta} + \frac{1}{2}\delta^2 S^2\frac{\partial^2 F}{\partial\delta^2} + (r-\delta)S\frac{\partial F}{\partial S} + (\kappa(a-\delta)-\lambda)\frac{\partial F}{\partial\delta} - \frac{\partial F}{\partial t} = 0 \quad (7)$$

The **analytical solution of this PDE** model is given by

$$F(S,\delta,t) = Sexp[-\delta\frac{1-e^{\kappa t}}{\kappa} + A(t)]$$

where $$A(t) = (r - \tilde{a} + \frac{1}{2}\frac{\sigma_2^2}{\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}t) + \frac{1}{4}\sigma_2^2\frac{1-e^{-2\kappa t}}{\kappa^2} +$$

$$+(\tilde{a}\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa})\frac{1-e^{-\kappa t}}{\kappa^2}$$

3. Boundary control of the multi-factor commodities price PDE

Next, the **multi-asset Black-Scholes PDE** is introduced

$$\frac{\partial V}{\partial t} = \sum_{i=1}^{N}\sum_{j=1}^{N}\rho\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{N} r S_i \frac{\partial V}{\partial S_i} - rV \quad (8)$$

Moreover, without loss of generality **the two-asset Black-Scholes PDE** is considered

$$\frac{\partial V}{\partial t} = \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial V}{\partial^2 S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial V}{\partial^2 S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV \quad (9)$$

The above **2-asset Black-Scholes PDE** is shown to be **equivalent** to the **2-factor commodities price PDE** that was described in Eq. 7

This is demonstrated through the **change of variables** $S_1 = S$ that is S_1 is equal to the **spot price**, $S_2 = \delta$ that is S_2 is equal to the **convenience yield** and after the coefficients of the three last partial derivative terms appearing in the right of Eq. (9) are suitably modified to arrive at the form:

$$\frac{\partial V}{\partial t} = \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial V}{\partial^2 S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial V}{\partial^2 S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + (r - S_2)\frac{\partial V}{\partial S_1} + (\kappa(a - S_2) - \lambda)\frac{\partial V}{\partial S_2} \quad (10)$$

3. Boundary control of the multi-factor commodities price PDE

Semi-discretization and the **finite differences method** is applied to the PDE model of Eq. (10). To this end the partial derivatives appearing in Eq. (9) are computed as follows

$$\frac{\partial V}{\partial S_1}=\frac{V(S_{1,i+1},s_{2,j})-V(S_{1,i},s_{2,j})}{\Delta S_1} \quad (11)$$

$$\frac{\partial^2 V}{\partial^2 S_1}=\frac{V(S_{1,i+1},S_{2,j})-2V(S_{1,i},S_{2,j})+V(S_{1,i-1},S_{2,j})}{\Delta {S_1}^2} \quad (12)$$

$$\frac{\partial V}{\partial S_2}=\frac{V(S_{1,i},s_{2,j+1})-V(S_{1,i},s_{2,j})}{\Delta S_2} \quad (13)$$

$$\frac{\partial^2 V}{\partial^2 S_2}=\frac{V(S_{1,i},S_{2,j+1})-2V(S_{1,i},S_{2,j})+V(S_{1,i},S_{2,j-1})}{\Delta {S_1}^2} \quad (14)$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2}=\frac{V(S_{1,i+i},S_{2,j+i})-V(S_{1,i+1},S_{2,j})-V(S_{1,i},S_{2,j+1})+V(S_{1,i},S_{2,j})}{\Delta S_1 \Delta S_2} \quad (15)$$

Using the previous semi-discretization, for grid point *(i,j)* it holds

$$\begin{aligned}\frac{\partial V(S_{1,i},S_{2,j})}{\partial t} &= \frac{1}{2}\sigma_1^2 S_{1,i}^2\left[\frac{V(S_{1,i+1},S_{2,j})-2V(S_{1,i},S_{2,j})+V(S_{1,i-1},S_{2,j})}{\Delta S_1^2}\right]+ \\ &\frac{1}{2}\sigma_2^2 S_{2,j}^2\left[\frac{V(S_{1,i},S_{2,j+1})-2V(S_{1,i},S_{2,j})+V(S_{1,i},S_{2,j-1})}{\Delta S_1^2}\right]+ \\ &\rho\sigma_1\sigma_2 S_{1,i}S_{2,j}\left[\frac{V(S_{1,i+1},S_{2,j+1})-V(S_{1,i+1},S_{2,j})-V(S_{1,i},S_{2,j+1})+V(S_{1,i},S_{2,j})}{\Delta S_1 \Delta S_2}\right]+ \\ &(r-S_{2,j})\left[\frac{V(S_{1,i+1},S_{2,j})-V(S_{1,i},S_{2,j})}{\Delta S_1}\right]+(\kappa(a-S_{2,j})-\lambda)\left[\frac{V(S_{1,i},S_{2,j+1})-V(S_{1,i},S_{2,j})}{\Delta S_2}\right]\end{aligned} \quad (17)$$

3. Boundary control of the multi-factor commodities price PDE

The **boundary conditions of the PDE** are taken to be

$$V_{i,0} \neq 0 \text{ only if } i = 1 \qquad V_{0,j} \neq 0 \text{ only if } j = 1$$

$$V(i,j) = ct \text{ (constant) if } i > N \text{ or } j > N$$

Considering $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, N$ the commodity's values at the grid points (i, j) are denoted as $V_{i,j}$. Using this notation, the **semi-discretized model of the PDE** takes the following form

At grid point $i = 1$ and $j = 1$

$$\begin{array}{c} \frac{\partial V_{1,1}}{\partial t} = \frac{1}{2}\sigma_1^2 S_{1,1}^2 [\frac{V_{2,1} - 2V_{1,1} + V_{0,1}}{\Delta S_1^2}] + \\ \frac{1}{2}\sigma_2^2 S_{2,1}^2 [\frac{V_{1,2} - 2V_{1,1} + V_{1,0}}{\Delta S_2^2}] + \rho\sigma_1\sigma_2 S_{1,1} S_{2,1} [\frac{V_{2,2} - V_{2,1} - V_{1,2} + V + 1,1}{\Delta S_1 \Delta S_2}] + \\ (r - S_{2,1})[\frac{V_{2,1} - V_{1,1}}{\Delta S_1}] + (\kappa(a - S_{2,1}) - \lambda)[\frac{V_{1,2} - V_{1,1}}{\Delta S_2}] \end{array} \qquad (18)$$

At grid point $i > 1$ and $j > 1$ it holds

$$\begin{array}{c} \frac{\partial V_{i,j}}{\partial t} = \frac{1}{2}\sigma_1^2 S_{1,i}^2 [\frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{\Delta S_1^2}] + \\ \frac{1}{2}\sigma_2^2 S_{2,j}^2 [\frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta S_2^2}] + \rho\sigma_1\sigma_2 S_{1,i} S_{2,j} [\frac{V_{i+1,j+1} - V_{i+1,j+1} - V_{i,j+1} + V + i,i}{\Delta S_1 \Delta S_2}] + \\ (r - S_{2,j})[\frac{V_{i+1,j} - V_{i,j}}{\Delta S_1}] + (\kappa(a - S_{2,j}) - \lambda)[\frac{V_{i,j+1} - V_{i,j}}{\Delta S_2}] \end{array} \qquad (19)$$

3. Boundary control of the multi-factor commodities price PDE

Next, the following **state vector variables** are defined

$$x_{(i-1)N+j} = V_{i,j},\ i = 1,2,\cdots,N \text{ and } j = 1,2,\cdots,N.$$

Using this notation of state variables Eq. (18) becomes

$$\begin{array}{c}\dot{x}_1 = \frac{1}{2}\sigma_1^2 S_{1,1}^2[\frac{x_{N+1}-2x_1}{\Delta S_1^2}] + \frac{1}{2}\sigma_2^2 S_{2,1}^2[\frac{x_2-2x_1}{\Delta S_1^2}]+ \\ \rho\sigma_1\sigma_2 S_{1,1}S_{1,2}[\frac{x_{N+2}-x_2-x_{N+1}+x_1}{\Delta S_1 \Delta S_2}] + (r-S_{2,1})[\frac{x_{N+1}-x_1}{\Delta S_1}] + (\kappa(a-S_{2,1})-\lambda)[\frac{x_2-x_1}{\Delta S_2}]+ \\ [\frac{1}{2}\sigma_1^2 S_{1,1}^2 \frac{V_{0,1}}{\Delta S_1^2} + \frac{1}{2}\sigma_2^2 S_{2,1}^2 \frac{V_{1,0}}{\Delta S_1^2}]\end{array} \quad (20)$$

Thus, by defining the **control input associated with the boundary conditions** as

$u = [V_{0.1}, V_{1,0}]^T$ and $c_1 u = [\frac{1}{2}\sigma_1^2 S_{1,1}^2 \frac{V_{0,1}}{\Delta S_1^2} + \frac{1}{2}\sigma_2^2 S_{2,1}^2 \frac{V_{1,0}}{\Delta S_1^2}]$

one obtains

$$\dot{x}_1 = f_1(x) + c_1 u \quad (21)$$

Equivalently, one has that

3. Boundary control of the multi-factor commodities price PDE

$$
\begin{aligned}
\dot{x}_{(i-1)N+j} = {} & \tfrac{1}{2}\sigma_1^2 S_{1,i}^2 [\tfrac{x_{iN+j}-2x_{(i-1)N+j}+x_{(i-2)N+j}}{\Delta S_1^2}]+ \\
& \tfrac{1}{2}\sigma_2^2 S_{2,j}^2 [\tfrac{x_{(i-1)N+(j+1)}-2x_{(i-1)N+j}+x_{(i-1)N+(j-1)}}{\Delta S_2^2}]+ \\
& \rho\sigma_1\sigma_2 S_{1,i} S_{2,j} [\tfrac{x_{(i)N+(j+1)}-x_{(i-1)N+(j+1)}-x_{iN+j}+x_{(i-1)N+j}}{\Delta S_1 \Delta S_2}]+ \\
& (r-S_{2,j})[\tfrac{x_{iN+j}-x_{(i-1)N+j}}{\Delta S_1}]+(\kappa(a-S_{2,j})-\lambda)[\tfrac{x_{(i-1)N+(j+1)}-x_{(i-1)N+j}}{\Delta S_2}]
\end{aligned}
\qquad (22)
$$

Eq. (22) can be also written as

$$
\begin{aligned}
\dot{x}_{(i-1)N+j} = {} & \tfrac{1}{2}\sigma_1^2 S_{1,i}^2 [\tfrac{x_{iN+j}-2x_{(i-1)N+j}+x_{(i-2)N+j}}{\Delta S_1^2}]+ \\
& \tfrac{1}{2}\sigma_2^2 S_{2,j}^2 [\tfrac{x_{(i-1)N+(j+1)}-2x_{(i-1)N+j}}{\Delta S_2^2}]+ \\
& \rho\sigma_1\sigma_2 S_{1,i} S_{2,j} [\tfrac{x_{(i)N+(j+1)}-x_{(i-1)N+(j+1)}-x_{iN+j}+x_{(i-1)N+j}}{\Delta S_1 \Delta S_2}]+ \\
& (r-S_{2,j})[\tfrac{x_{iN+j}-x_{(i-1)N+j}}{\Delta S_1}]+(\kappa(a-S_{2,j}-\lambda)[\tfrac{x_{(i-1)N+(j+1)}-x_{(i-1)N+j}}{\Delta S_2}]+ \\
& [\tfrac{1}{2}\sigma_2^2 S_{2,j}^2 \tfrac{1}{\Delta S_2^2}]x_{(i-1)N+(j-1)}
\end{aligned}
\qquad (23)
$$

Eq. (23) can be also written as

$$
\dot{x}_{(i-1)N+j} = f_{(i-1)N+j}(x) + c_{(i-1)N+j}x_{(i-1)N+(j-1)} \qquad (24)
$$

where

3. Boundary control of the multi-factor commodities price PDE

$$
\begin{array}{c}
f_{(i-1)N+j}(x) = \frac{1}{2}\sigma_1^2 S_{1,i}^2 [\frac{x_{iN+j} - 2x_{(i-1)N+j} + x_{(i-2)N+j}}{\Delta S_1^2}] + \\
\frac{1}{2}\sigma_2^2 S_{2,j}^2 [\frac{x_{(i-1)N+(j+1)} - 2x_{(i-1)N+j}}{\Delta S_2^2}] + \\
\rho\sigma_1\sigma_2 S_{1,i} S_{2,j} [\frac{x_{(i)N+(j+1)} - x_{(i-1)N+(j+1)} - x_{iN+j} + x_{(i-1)N+j}}{\Delta S_1 \Delta S_2}] + \\
(r - S_{2,j})[\frac{x_{iN+j} - x_{(i-1)N+j}}{\Delta S_1}] + (\kappa(a - S_{2,j} - \lambda)[\frac{x_{(i-1)N+(j+1)} - x_{(i-1)N+j}}{\Delta S_2}]
\end{array}
\quad (25)
$$

$$
c_{(i-1)N+j} = [\frac{1}{2}\sigma_2^2 S_{2,j}^2 \frac{1}{\Delta S_2^2}]
$$

and

$$
c_{(i-1)N+j} = [\frac{1}{2}\sigma_2^2 S_{2,j}^2 \frac{1}{\Delta S_2^2}] \quad (26)
$$

Considering that $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, N$ there are ***N*²** **state-space equations**. Thus, the dynamics of the PDE model is written as

$$
\begin{array}{c}
\dot{x}_{N^2} = f_{N^2}(x) + c_{N^2} x_{N^2-1} \\
\dot{x}_{N^2-1} = f_{N^2-1}(x) + c_{N^2-1} x_{N^2-2} \\
\cdots \quad \cdots \\
\dot{x}_{(i-1)N+j} = f_{(i-1)N+j} + c_{(i-1)N+j} x_{(i-1)N+(j-1)} \\
\cdots \quad \cdots \\
\dot{x}_2 = f_2(x) + c_2 x_1 \\
\dot{x}_1 = f_1(x) + c_1 u
\end{array}
\quad (27)
$$

4. Flatness-based control of the multi-factor commodities price PDE

First, it can be proven that **the state-space description of the commodities price PDE**, given in Eq. (27), **is a differentially flat one**, with flat output $y = x_{N^2}$

Solving the i-th row of the state space model with respect to x_{i+1} one finds that state variables x_{i+1} is a differential function of the flat output y. Moreover, from the last row of Eq. (27) it holds that u is a function of the flat output and its derivatives. Next, the following virtual control inputs are defined

$$\alpha_1 = x_{N^2-1}, \quad \alpha_2 = x_{N^2-2}, \quad \cdots \quad \alpha_{N^2-(i-1)N-(j-1)} = x_{(i-1)N+(j-1)}, \quad \cdots \quad \alpha_{N^2-1} = x_1 \tag{28}$$

Using the virtual control inputs of Eq. (28) in the state-space model of Eq. (27) one gets

$$\begin{gathered}
\dot{x}_{N^2} = f_{N^2}(x) + c_{N^2}\alpha_1 \\
\dot{x}_{N^2-1} = f_{N^2-1}(x) + c_{N^2-1}\alpha_2 \\
\cdots \; \cdots \\
\dot{x}_{(i-1)N+j} = f_{(i-1)N+j} + c_{(i-1)N+j}\alpha_{N^2-(i-1)N-(j-1)} \\
\cdots \; \cdots \\
\dot{x}_2 = f_2(x) + c_2\alpha_{N^2-1} \\
\dot{x}_1 = f_1(x) + c_1 u
\end{gathered} \tag{29}$$

4. Flatness-based control of the multi-factor commodities price PDE

By examining **independently each nonlinear ODE** of the previous state-space description of Eq. (29) and by defining as local flat output for the *i-th* ODE the state variable x_i it can be shown that the ***i-th*** **row of the state-space description** stands again for a **differentially flat system**.

- Actually, one has now N^2 subsystems, each one of them related to a row of the state-space model and the **local flat outputs for these subsystems** are

$$Y = [x_1, x_2, \cdots, x_{(i-1)N+j}, \cdots, x_{N^2-1}, x_{N^2}] \qquad (30)$$

- From the **i-th row of the state-space model** it can be seen that the virtual control input α_i is a **differential function** of the local flat output x_i, which shows again that **the i-th subsystem**, if **independently examined**, is also **differentially flat**.

One can find **the values that the virtual control inputs** α_i should have, so as to eliminate the tracking error for each one of the subsystems \ that are obtained from the per-row decomposition

4. Flatness-based control of the multi-factor commodities price PDE

Virtual control inputs stabilizing the commodities price PDE:

$$\alpha_1^* = \frac{1}{c_{N^2}}[\dot{x}_N^* - f_{N^2}(x) - k_{p_1}(x_{N^2} - x_{N^2}^*)] \quad (31)$$

$$\alpha_2^* = \frac{1}{c_{N^2-1}}[\dot{\alpha}_1^* - f_{N^2-1}(x) - k_{p_2}(x_{N^2-1} - \alpha_1^*)] \quad (32)$$

$$\cdots \quad \cdots$$

$$\alpha_{N^2-(i-1)N-(j-1)} = \frac{1}{c_{(i-1)N+j}}[\dot{\alpha}_{N^2-(i-1)N-(j-1)-1}^* - f_{(i-1)N+j}(x) - \\ -k_{pN^2-(i-1)N-(j-1)}(x_{(i-1)N+j} - \alpha_{N^2-(i-1)N-(j-1)-1}^*)] \quad (33)$$

$$\alpha_{N^2-(i-1)N-(j-1)} = \frac{1}{c_{(i-1)N+j}}[\dot{\alpha}_{N^2-(i-1)N-(j-1)-1}^* - f_{(i-1)N+j}(x) - \\ -k_{pN^2-(i-1)N-(j-1)}(x_{(i-1)N+j} - \alpha_{N^2-(i-1)N-(j-1)-1}^*)] \quad (34)$$

$$\alpha_{N^2-1}^* = \frac{1}{c_2}[\dot{\alpha}_{N^2-2}^* - f_2(x) - k_{pN^2-1}(x_2 - \alpha_{N^2-2}^*)] \quad (35)$$

$$\cdots \quad \cdots$$

$$u = \frac{1}{c_1}[\dot{\alpha}_{N^2-1}^* - f_1(x) - k_{pN^2}(x_1 - \alpha_{N^2-1}^*)] \quad (36)$$

5. Stability analysis of the control loop of the multi-factor commodities price PDE

The **dynamics of the multi-factor commodities PDE system** has been shown to be

$$\dot{x}_{N^2} = f_{N^2}(x) + c_{N^2}\alpha_1 \quad (37)$$

$$\dot{x}_{N^2-1} = f_{N^2-1}(x) + c_{N^2-1}\alpha_2 \quad (38)$$

$$\cdots \quad \cdots$$

$$\dot{x}_{(i-1)N+j} = f_{(i-1)N+j} + c_{(i-1)N+j}\alpha_{N^2-(i-1)N-(j-1)} \quad (39)$$

$$\cdots \quad \cdots$$

$$\dot{x}_2 = f_2(x) + c_2\alpha_{N^2-1} \quad (40)$$

$$\dot{x}_1 = f_1(x) + c_1 u \quad (41)$$

From Eq. 36 and Eq. 41 one gets

$$\begin{array}{c}\dot{x}_1 = f_1(x) + c_1\frac{1}{c_1}[\dot{\alpha}^*_{N^2-1} - f_1(x) - k_{p_{N^2}}(x_1 - \alpha^*_{N^2-1})] \Rightarrow \\ (\dot{x}_1 - \dot{\alpha}^*_{N^2-1}) + k_{p_{N^2}}(x_1 - \alpha^*_{N^2-1})] = 0\end{array} \quad (42)$$

5. Stability analysis of the control loop of the multi-factor commodities price PDE

By defining $z_1 = x_1 - \alpha^*_{N^2\_1}$ and taking $k_{p_{N^2}} > 0$ one obtains

$$\dot{z}_1 + k_{p_{N^2}} z_1 = 0 \Rightarrow lim_{t\to\infty} z_1 = 0$$
$$\Rightarrow lim_{t\to\infty} x_1 = \alpha^*_{N^2\_1} \Rightarrow lim_{t\to}\infty x_1 = x_1^* \quad (43)$$

From Eq. (35) and Eq. (40) one gets

$$\dot{x}_2 = f_2(x) + c_2 \tfrac{1}{c_2}[\dot{\alpha}^*_{N^2\_2} - f_2(x) - k_{p_{N^2\_1}}(x_2 - \alpha^*_{N^2\_2})] \Rightarrow$$
$$(\dot{x}_2 - \dot{\alpha}^*_{N^2\_2}) + k_{p_{N^2\_1}}(x_2 - \alpha^*_{N^2\_2}) = 0 \quad (44)$$

By defining $z_2 = x_2 - \alpha^*_{N^2\_1}$ and taking $k_{p_{N^2}} > 0$ one obtains

$$\dot{z}_2 + k_{p_{N^2}} z_2 = 0 \Rightarrow lim_{t\to\infty} z_2 = 0$$
$$\Rightarrow lim_{t\to\infty} x_2 = \alpha^*_{N^2\_2} \Rightarrow lim_{t\to}\infty x_2 = x_2^* \quad (45)$$

This procedure is also applied to the rest of the rows **of the PDE's state-space description**

5. Stability analysis of the control loop of the multi-factor commodities price PDE

From Eq. 34 and Eq. 39 one gets 46

$$\dot{x}_{(i-1)N+j} = f_{(i-1)N+j}(x) + c_{(i-1)N+j}\frac{1}{c_{(i-1)N+j}}[\dot{\alpha}^*_{N^2-(i-1)N-(j-1)-1} - f_{(i-1)N+j}(x) - k_{p_{N^2-(i-1)N-(j-1)}}(x_{(i-1)N+j} - \alpha^*_{N^2-(i-1)N-(j-1)-1})] \Rightarrow$$

$$(\dot{x}_{(i-1)N+j} - \dot{\alpha}^*_{N^2-(i-1)N-(j-1)-1}) + k_{p_{N^2-(i-1)N-(j-1)}}(x_{(i-1)N+j} - \alpha^*_{N^2-(i-1)N-(j-1)-1}) = 0$$

By defining $z_{N^2-(i-1)N-(j-1)} = x_{(i-1)N+j} - \alpha^*_{N^2-(i-1)N-(j-1)-1} - 1$ and taking $k_{p_{N^2-(i-1)N-(j-1)}} > 0$ one obtains

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$$\dot{z}_{N^2-(i-1)N-(j-1)} + k_{p_{N^2-(i-1)N-(j-1)}} z_{N^2-(i-1)N-(j-1)} = 0 \Rightarrow lim_{t\to\infty} z_{N^2-(i-1)N-(j-1)} = 0$$
$$\Rightarrow lim_{t\to\infty} x_{(i-1)N+j} = \alpha^*_{N^2-(i-1)N-(j-1)-1} \Rightarrow lim_{t\to\infty} x_{(i-1)N+j} = x^*_{(i-1)N+j}$$

From Eq. 33 and Eq. 38 one gets

$$\dot{x}_{N^2-1} = f_{N^2-1}(x) + c_{N^2-1}\frac{1}{c_{N^2-1}}[\dot{\alpha}^*_1] - f_{N^2-1}(x) - k_{p_2}(x_{N^2-1} - \alpha^*_1) \Rightarrow$$
$$(\dot{x}_{N^2-1} - \dot{\alpha}^*_1) + k_{p_2}(x_{N^2-1} - \dot{\alpha}^*_1)) = 0$$

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5. Stability analysis of the control loop of the multi-factor commodities price PDE

By defining $z_{N^2-1} = x_{N^2-1} - \alpha_1$ and taking $k_{p_2} > 0$ one obtains

$$\dot{z}_{N^2-1} + k_{p_2} z_{N^2-1} = 0 \Rightarrow lim_{t\to\infty} z_{N^2-1} = 0 \Rightarrow$$
$$lim_{t\to\infty} x_{N^2-1} = \alpha_1^* \Rightarrow lim_{t\to\infty} x_{N^2-1} = x^*_{N^2-1} \qquad (49)$$

From Eq. (32) and Eq. (37) one gets

$$\dot{x}_{N^2} = f_{N^2}(x) + c_{N^2}\frac{1}{c_{N2}}[\dot{x}^*_N] - f_{N^2}(x) - k_{p1}(x_{N^2} - x^*_{N^2}) \Rightarrow$$
$$(\dot{x}_{N^2} - \dot{x}^*_{N^2}) + k_{p1}(x_{N^2} - \dot{x}^*_{N^2})) = 0 \qquad (50)$$

By defining $z_{N^2} = x_{N^2} - x^*_{N^2}$ and taking $k_{p_1} > 0$ one obtains

$$\dot{z}_{N^2} + k_{p_1} z_{N^2} = 0 \Rightarrow lim_{t\to\infty} z_{N^2} = 0 \Rightarrow$$
$$lim_{t\to\infty} x_{N^2} = x^*_{N^2} \qquad (51)$$

Through this procedure, it is proven that **the tracking error** for the **individual control loops** into which the PDE model is decomposed **converges asymptotically to 0**.

5. Stability analysis of the control loop of the multi-factor commodities price PDE

From the previous analysis one can also demonstrate the **stability of the control loop** by **applying the Lyapunov method**. It holds that

$$\begin{gathered}\dot{z}_{N^2}+k_{p_1}z_{N^2}=0\\ \dot{z}_{N^2-1}+k_{p_2}z_{N^2-1}=0\\ \cdots\ \cdots\\ \dot{z}_{N^2-(i-1)N-(j-1)}+k_{p_{N^2-(i-1)N-(j-1)}}z_{N^2-(i-1)N-(j-1)}=0\\ \cdots\ \cdots\\ \dot{z}_2+k_{p_{N^2-1}}z_2=0\\ \dot{z}_1+k_{p_{N^2}}z_1=0\end{gathered} \quad (52)$$

The following **Lyapunov function** is defined

$$V=\sum_{i=1}^{N^2}\frac{1}{2}z_i^2 \quad (53)$$

Setting, $k_{p_i}>0 \text{ for } i=1,2,\cdots,N^2$ the **first derivative of this Lyapunov function** is

$$\begin{gathered}\dot{V}=\sum_{i=1}^{N^2}\frac{1}{2}2z_i\dot{z}_i\Rightarrow\dot{V}=\sum_{i=1}^{N^2}z_i(-k_{p_i}z_i)\Rightarrow\\ \dot{V}=-\sum_{i=1}^{N^2}k_{p_i}z_i^2\Rightarrow\dot{V}<0\end{gathered} \quad (54)$$

The above result **confirms the asymptotic stability** of the **multi-factor commodities price PDE** control loop,that has been based on differential flatness theory.

6. Simulation tests

The numerical simulation experiments have confirmed the theoretical findings. It has been shown that by applying the proposed control method, **the multi-factor commodities PDE** dynamics can be modified so as to **converge to the desirable reference setpoints**.

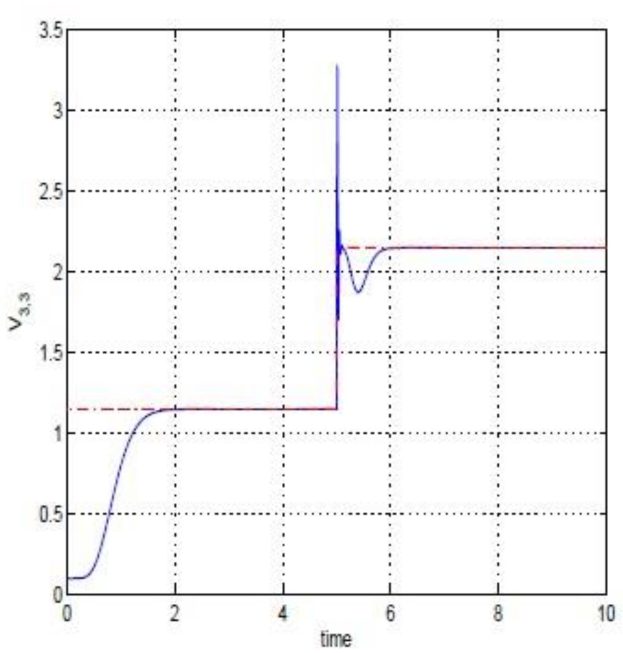

Fig. 1 Setpoint 1: Tracking of reference setpoint (dashed red line) by the PDE system (blue line) at the final grid point

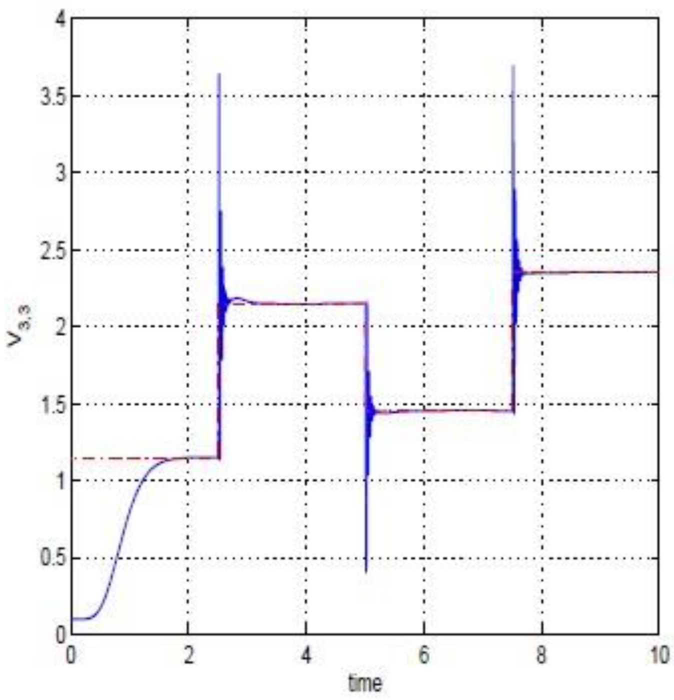

Fig. 2 Setpoint 2: Tracking of reference setpoint (dashed red line) by the PDE system (blue line) at the final grid point

6. Simulation tests

The **control input that succeeds stabilization** of the **Commodities Price PDE** has a moderate range of variation. The accuracy of tracking of the reference setpoints was quite satisfactory.

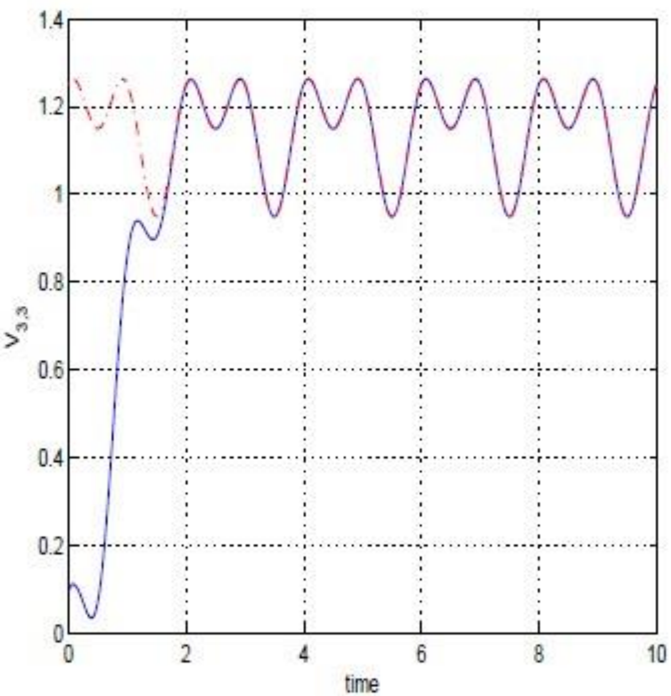

Fig. 3 Setpoint 3: Tracking of reference setpoint (dashed red line) by the PDE system (blue line) at the final grid point

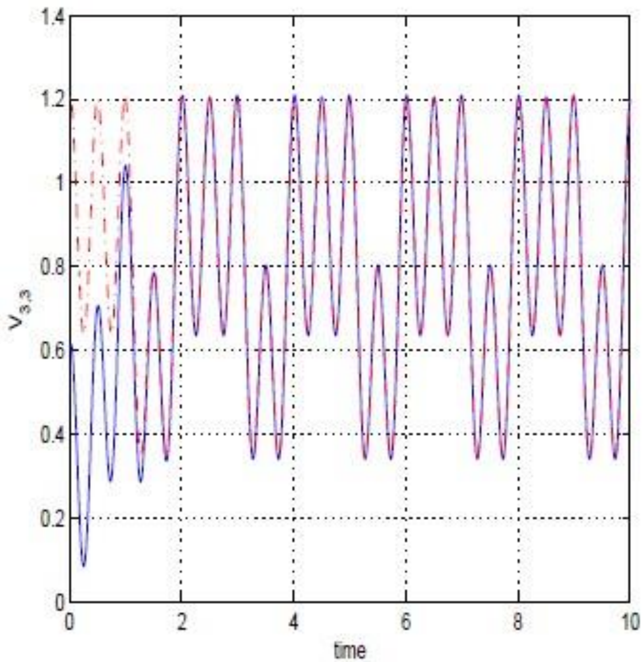

Fig. 4 Setpoint 4: Tracking of reference setpoint (dashed red line) by the PDE system (blue line) at the final grid point

6. Simulation tests

The proposed method shows that **stabilization of financial systems dynamics** is **possible through feedback control**

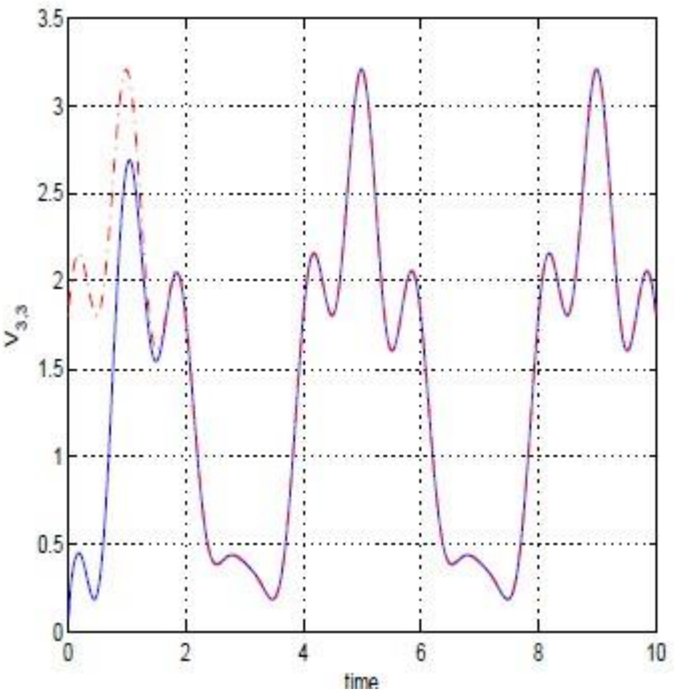

Fig. 5 Setpoint 5: Tracking of reference setpoint (dashed red line) by the PDE system (blue line) at the final grid point

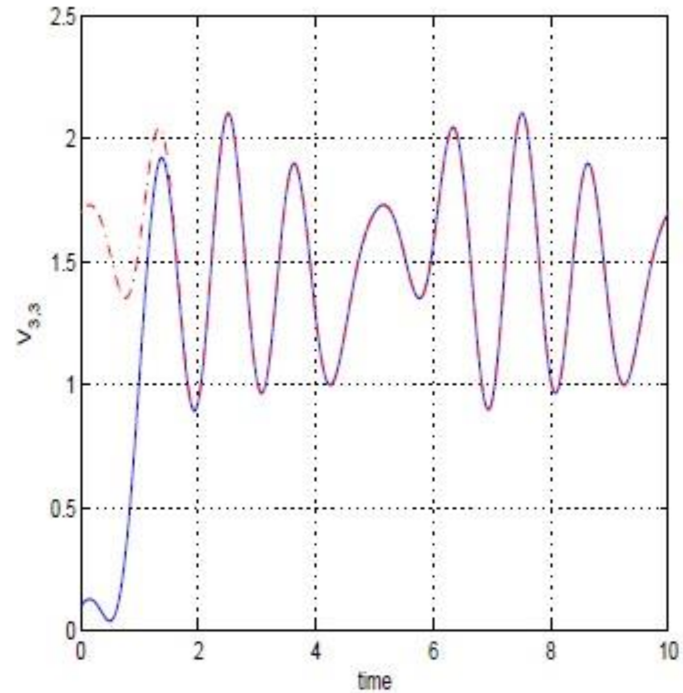

Fig. 6 Setpoint 6: Tracking of reference setpoint (dashed red line) by the PDE system (blue line) at the final grid point

8. Conclusions

- The **Commodities Price PDE** has been shown to be equivalent to a **multi-asset 2D Black-Scholes PDE**.

- **Following semi-discretization and a finite differences scheme**, the **Commodities Price PDE model** has been **decomposed into an equivalent set of nonlinear ordinary differential equations (ODEs)** and a **state-space model** has been obtained.

- Next, it has been proven that **each one of the aforementioned ODEs** stands for a **differentially flat subsystem**. This enables to compute for each ODE subsystem a **virtual control input** which linearizes its dynamics and eliminates the output's tracking error.

- From the state equations that constitute the last subsystem one can find **the boundary condition that also stands for the control input** to the Commodities Price PDE model.

- To compute the **boundary control input of the Commodities Price PDE model** one has to **use recursively all virtual control inputs** which are applied to the previously mentioned ODE subsystems. The computation of control inputs moves from the last to the first ODE.

- Consequently, by **tracing the rows of the state-space model backwards**, **the boundary control input** that stabilizes the Commodities Price PDE is obtained.

- The **asympotic stability** of the control metho**d has been proven.**