Nonlinear control and estimation for USVs and AUVs: advances in autonomous navigation and optimized propulsion

Part A: Autonomous navigation

Gerasimos Rigatos
Electrical and Computer Eng., Ph.D.
Research Director

Unit of Industrial Automation
Industrial Systems Institute
26504, Rion Patras, Greece

email: grigat@ieee.org

I . Outline

● Autonomous navigation of USVs and AUVs relies on the solution of the associated nonlinear control and state estimation problems

● The main approaches followed towards the solution of nonlinear control problem are as follows: (i) **control with global linearization** methods (ii) **control with approximate (asymptotic) linearization** methods (iii) **control with Lyapunov theory methods** (adaptive control methods) when the dynamic model of the USVs and AUVs is unknown

● The main approaches followed towards the solution of the nonlinear state estimation problems are as follows: (i) state estimation with methods global linearization (ii) state estimation with methods of approximate (asymptotic) linearization

● Factors of major importance for the control loop of USVs and AUVs, in autonomous navigation problems, are as follows (i) global stability conditions for the related nonlinear control scheme (ii) global stability conditions for the related nonlinear state estimation scheme (iii) global asymptotic stability for the joint control and state estimation scheme

II . Nonlinear control and state estimation with global linearization

● To this end the differential flatness control theory is used

● The method can be applied to all nonlinear systems which are subject to an input-output linearization and actually such systems posses the property of differential flatness

● The state-space description for the dynamic model of the USVs and AUVs is transformed into a more compact form that is input-output linearized. This is achieved after defining the system's flat outputs

● A system is differentially flat if the following two conditions hold: (i) all state variables and control inputs of the system can be expressed as differential functions of its flat outputs (ii) the flat outputs of the system and their time-derivatives are differentially independent, which means that they are not connected through a relation having the form of an ordinary differential equation

● With the applications of change of variables (diffeomorphisms) that rely on the differential flatness property (i), the state-space description of the USVs and AUVs is written into the linear canonical form. For the latter state-space description it is possible to solve both the control and the state estimation problem for USVs and AUVs, and to achieve autonomous navigation..

III . Nonlinear control and state estimation with approximate linearization

● To this end the theory of optimal H-infinity control and the theory of optimal H-infinity state estimation are used

● The nonlinear state-space description of the USVs and AUVs undergoes approximate linearization around a temporary operating point which is updated at each iteration of the control and state estimation algorithm

● The linearization relies on first order Taylor series expansion around the temporary operating point and makes use of the computation of the associated Jacobian matrices

● The linearization error which is due to the truncation error of higher-order terms in the Taylor series expansion is considered to be a perturbation that is finally compensated by the robustness of the control algorithm

● For the linearized description of the state-space model an optimal H-infinity controller is designed. For the selection of the controller's feedback gains an algebraic Riccati equation has to be solved at each time step of the control algorithm

● Through Lyapunov stability analysis, the global stability properties of the control method are proven

● For the implementation of the optimal control method through the processing of measurements from a small number of sensors in the USVs and AUVs, the H-infinity Kalman Filter is used as a robust state estimator

IV . Nonlinear control and state estimation with Lyapunov methods

● By proving differential flatness properties for USVs and AUVs and by defining the associated flat outputs, a transformation of the USVs and AUVs state-space model into an equivalent input-output linearized form is achieved.

● The unknown dynamics of the USVs and AUVs is incorporated into the transformed control inputs of the system, which now appear in its equivalent input-output linearized state-space description

● The control problem for USVs and AUVs of unknown dynamics in now turned into a problem of indirect adaptive control. The computation of the control inputs of the system is performed simultaneously with the identification of the nonlinear functions which constitute its unknown dynamics.

● The estimation of the unknown dynamics of the USVs and AUVs is performed through the adaptation of neurofuzzy approximators. The definition of the learning parameters takes place through gradient algorithms of proven convergence, as demonstrated by Lyapunov stability analysis

● The Lyapunov stability method is the tool for selecting both the gains of the stabilizing feedback controller and the learning rate of the estimator of the unknown system's dynamics

● Equivalently through Lyapunov stability analysis the feedback gains of the state estimators of the USVs and AUVs are chosen. Such observers are included in the control loop so as to enable feedback control through the processing of a small number of sensor measurements

Example 1: Nonlinear control and state estimation using global linearization

1. Control of a 3-DOF underactuated USV

- The nonlinear model of the **underactuated vessel** is a **differentially flat** one. This model cannot be subjected to static feedback linearization, however it admits **dynamic feedback linearization** which means that the **system's state vector is extended** by including as additional state variables the control inputs and their derivatives.

- Next, using the **differential flatness properties** it is also proven that this model can be subjected to **input-output linearization** and can be transformed to an equivalent **canonical (Brunovsky) form**. Based on this the design of a state **feedback controller** is carried out enabling accurate manoeuvring and trajectory tracking.

- The **Derivative-free nonlinear Kalman Filter** is used as **disturbance observer** for dynamically identifying model uncertainty and external perturbation terms. .

- This nonlinear filter consists of the Kalman Filter's recursion on the linearized equivalent model of the vessel and of an inverse **nonlinear transformation based on the differential flatness** features of the system which enables to compute state estimates for the state variables of the initial nonlinear model.

- The redesign of the filter as a **disturbance observer** makes possible the estimation and **compensation** of additive **perturbation terms** affecting the vessel's model.

Example 1: Nonlinear control and state estimation using global linearization

2. Model of the underactuated vessel

• The underactuated vessel's model stems from the **generic ship's model**, after setting specific values for the elements of the inertia and Coriolis matrix and after reducing the number of the available control inputs.

• The **state-space equation** of the nonlinear underactuated vessel is

$$\dot{x} = ucos(\psi) - vsin(\psi)$$
$$\dot{y} = usin(\psi) + vcos(\psi)$$
$$\dot{\psi} = r$$
$$\dot{u} = v{\cdot}r + \tau_u$$
$$\dot{v} = -u{\cdot}r - \beta v$$
$$\dot{r} = \tau_r$$

x and *y* are the cartesian coordinates of the vessel

ψ is the orientation angle

u is the surge velocity

v is the sway velocity

r is the yaw rate

The **control inputs** are the surge force τ_u and the yaw torque τ_r

The underactuated vessel's model is also written in the **matrix form**

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} ucos(\psi) - vsin(\psi) \\ usin(\psi) + vcos(\psi) \\ r \\ vr \\ -ur - \beta v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_u \\ \tau_r \end{pmatrix}$$

Example 1: Nonlinear control and state estimation using global linearization

2. Model of the underactuated vessel

or equivalently, one has the description $\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{v}$

The system's state vector is denoted as $\tilde{x} = [x, y, \psi, u, v, r]^T$

while $f(\tilde{x}) \in R^{6\times 1}$ and $\tilde{g}(\tilde{x}) = [\tilde{g}_a, \tilde{g}_b] \in R^{6\times 2}$

while the control input is the vector $\tilde{v} = [\tau_u, \tau_r]^T$

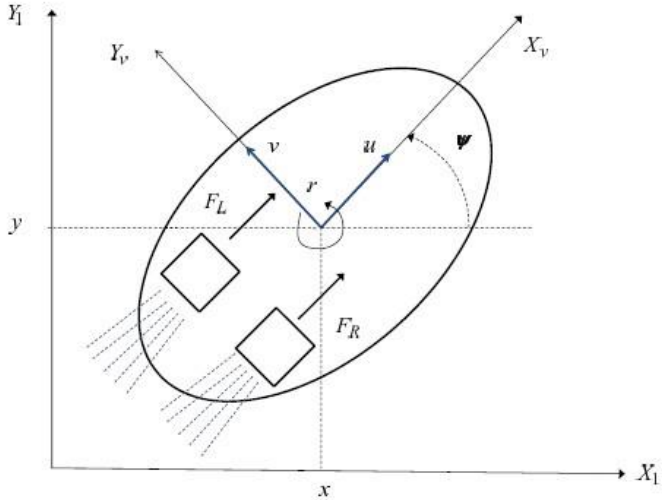

Fig. 1. Diagram of the underactuated hovercraft's kinematic model

Example 1: Nonlinear control and state estimation using global linearization

2. Model of the underactuated vessel

The system's state vector can be extended by including as additional state variables the control input τ_u and its first derivative $\dot{\tau}_u$.

The extended state-space description of the system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} ucos(\psi) - vsin(\psi) \\ usin(\psi) + vcos(\psi) \\ r \\ vr + z_1 \\ -ur - \beta v \\ 0 \\ z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \end{pmatrix}$$

or equivalently, one has the description $\dot{z} = f(z) + g(z)\tilde{v}$

The extended system's state vector is denoted as $z = [x, y, \psi, u, v, r, z_1, z_2]^T$.

Moreover, one has $f(z) \in R^{8 \times 1}$ and $g(z) = [g_a, g_b] \in R^{8 \times 2}$;

while the control input is the vector is $\tilde{v} = [\ddot{\tau}_u, \tau_r]^T$.

Example 1: Nonlinear control and state estimation using global linearization

3. Outline of differential flatness theory

- **Differential flatness theory** has been developed as a **global linearization control method** by M. Fliess (Ecole Polytechnique, France) and co-researchers .

- A dynamical system can be written in the ODE form $S_i(w,\dot{w},\ddot{w},\dots,w^{(i)}),\quad i=1,2,\dots,q$
  where $w^{(i)}$ stands for the i-th derivative of either a state vector element or of a control input

- The system is said to be **differentially flat** with respect to the **flat output**

$$y_i=\varphi(w,\dot{w},\ddot{w},\dots,w^{(a)}),\quad i=1,\dots,m \text{ where } y=(y_1,y_2,\dots,y_m)$$

if the following two conditions are satisfied

(i) There does not exist any differential relation of the form

$$R(y,\dot{y},\ddot{y},\dots,y^{(\beta)})=0$$

which means that **the flat output and its derivatives are linearly independent**

(ii) All system variables are **functions of the flat output and its derivatives**

$$w^{(i)}=\psi(y,\dot{y},\ddot{y},\dots,y^{(\gamma_i)})$$

Example 1: Nonlinear control and state estimation using global linearization

3. Outline of differential flatness theory

The proposed Lyapunov theory-based control method is based on the **transformation** of the nonlinear system's model into the **linear canonical form,** by exploiting the system's differential flatness properties

• **All single input nonlinear systems** are differentially flat and can be transformed into the linear canonical form

One has to define also which are the **MIMO nonlinear systems** which are differentially flat.

• Differential flatness holds for **MIMO nonlinear systems** that admit **static feedback linearization.**and which can be transformed into the linear canonical form through a change of variables (diffeomorphism) and feedback of the state vector.

• Differential flatness holds for **MIMO nonlinear models** that admit **dynamic feedback linearization**, This **is the case of specific underactuated robotic models.** In the latter case the state vector of the system is extended by considering as additional flat outputs some of the control inputs and their derivatives

• Finally, a more rare case is the so-called **Liouvillian systems.** These are systems for which differential flatness properties hold for part of their state vector while the non-flat state variables can be obtained by integration of the elements of the flat subsystem.

Example 1: Nonlinear control and state estimation using global linearization

4. Differential flatness of the model of the underactuated vessel

The flat output is the vector of the vessel's cartesian coordinates, that is

$$\tilde{y} = [y_1, y_2] = [x, y]$$

It holds that

$$\ddot{x} = \dot{u}cos(\psi) - u\cdot sin(\psi)\cdot\dot{\psi} - \dot{v}sin(\psi) - v\cdot cos(\psi)\dot{\psi}$$
$$\ddot{y} = \dot{u}sin(\psi) + u\cdot cos(\psi)\cdot\dot{\psi} + \dot{v}cos(\psi) - v\cdot cos(\psi)\dot{\psi} \quad (1)$$

Moreover, it holds that

$$\ddot{x} + \beta\dot{x} = cos(\psi)(\dot{u} - v\dot{\psi} + \beta u) + sin(\psi)(-u\dot{\psi} - \dot{v} - \beta v)$$
$$\ddot{y} + \beta\dot{y} = cos(\psi)(\dot{v} + u\dot{\psi} + \beta v) + sin(\psi)(-v\dot{\psi} + \dot{u} + \beta u) \quad (2)$$

Using Eq. (1) and Eq. (2), and after computing that

$$u\dot{\psi} + \dot{v} + \beta v = u\cdot r - ur - \beta v + \beta v = 0$$
$$\dot{u} - v\dot{\psi} + \beta u = vr + \tau_u - vr + \beta u = \tau_u + \beta u$$

one obtains that

$$\frac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}} = \frac{cos(\psi)0+sin(\psi)(\tau_u+\beta u)}{cos(\psi)(\tau_u+\beta u)-sin(\psi)0} \Rightarrow$$

$$\frac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}} = tan(\psi) \Rightarrow \psi = tan^{-1}\left(\frac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}}\right) \quad (3)$$

Example 1: Nonlinear control and state estimation using global linearization

4. Differential flatness of the model of the underactuated vessel

Through Eq. (3) it is proven that the state variable ψ (heading angle of the vessel) is a function of the flat output and of its derivatives.

From Eq. (1) one also has that

$$(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2=(\tau_u+\beta u)^2 \quad (4)$$

Moreover, it holds that

$$\begin{aligned}\dot{x}(\ddot{x}+\beta\dot{x})&=(ucos(\psi)-vsin(\psi))cos(\psi)(\tau_u+\beta u)\\ \dot{y}(\ddot{y}+\beta\dot{y})&=(usin(\psi)+vcos(\psi))sin(\psi)(\tau_u+\beta u)\end{aligned} \quad (5)$$

while using Eq. (4) and after intermediate computations one finally obtains

$$\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})=u(\tau_u+\beta u) \quad (6)$$

which finally gives

$$u=\frac{\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}} \quad (7)$$

It also holds that

Example 1: Nonlinear control and state estimation using global linearization

4. Differential flatness of the model of the underactuated vessel

$$\dot{y}\ddot{x}-\dot{x}\ddot{y} = (usin(\psi)+vcos(\psi))(\dot{u}cos(\psi)-usin(\psi)\dot{\psi}-\dot{v}sin(\psi)-vcos(\psi)\dot{\psi})-(ucos(\psi)-vsin(\psi))(\dot{u}sin(\psi)+ucos(\psi)\dot{\psi}+\dot{v}cos(\psi)-vsin(\psi)\dot{\psi})$$

which after intermediate computations and substitution of the derivative variables gives

$$\dot{y}\ddot{x}-\dot{x}\ddot{y}=v(\beta u+\tau_u) \quad (8)$$

From Eq. (8) and Eq. (4) one gets

$$v=\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\sqrt{(\ddot{x}+\beta\dot{x})^2+\ddot{y}+\beta\dot{y})^2}} \quad (9)$$

From the state-space equations it holds that

$$r=\dot{\psi} \quad (10)$$

and using Eq. (10) one also has that r is a differential function of the flat output

Example 1: Nonlinear control and state estimation using global linearization

4. Differential flatness of the model of the underactuated vessel

This can be also confirmed analytically. Indeed from Eq (3) it holds

$$\frac{cos^2(\psi)\dot{\psi}+sin^2(\psi)\dot{\psi}}{cos^2(\psi)} = \frac{(y^{(3)}+\beta\ddot{\psi})(\ddot{x}+\beta\dot{x})-(\ddot{y}+\beta\dot{y})(x^{(3)}+\beta\ddot{x})}{(\ddot{x}+\beta\dot{x})^2} \quad (11)$$

which also gives

$$\frac{\dot{\psi}}{cos^2(\psi)} = \frac{(y^{(3)}+\beta\ddot{\psi})(\ddot{x}+\beta\dot{x})-(\ddot{y}+\beta\dot{y})(x^{(3)}+\beta\ddot{x})}{(\ddot{x}+\beta\dot{x})^2} \quad (12)$$

while also using that

$$\frac{1}{\cos^2\psi} = tan^2(\psi)+1 \quad (13)$$

one obtains that

$$\cos^2\psi = \frac{(\ddot{x}+\beta\dot{x})^2}{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2} \quad (14)$$

Thus, from Eq. (12) and Eq. (10) one has

$$r=\dot{\psi} \Rightarrow r = cos^2(\psi)\frac{(y^{(3)}+\beta\ddot{\psi})(\ddot{x}+\beta\dot{x})-(\ddot{y}+\beta\dot{y})(x^{(3)}+\beta\ddot{x})}{(\ddot{x}+\beta\dot{x})^2} \quad (15)$$

Equivalently, from the extended state-space equations of the system one has that

Example 1: Nonlinear control and state estimation using global linearization

4. Differential flatness of the model of the underactuated vessel

$$\tau_u = \dot{u} - v \cdot r \Rightarrow \tau_u = \frac{d}{dt}\left\{\frac{\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}\right\} -$$

$$- \frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}} \cdot \frac{y^{(3)}(\ddot{x}+\beta\dot{x})-x^{(3)}(\ddot{y}+\beta\dot{y})-\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})}{(\ddot{x}+\beta\dot{x})^2+\ddot{y}+\beta\dot{y})^2} \quad (16)$$

which after intermediate operations gives $\tau_u = \frac{\ddot{x}(\ddot{x}+\beta\dot{x})+\ddot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}$ (17)

Finally, using that the control input $\tau_r = \dot{r}$ this implies also that τ_r is a differential function of the flat output

The above can be also shown analytically

$$\tau_r = \dot{r} \Rightarrow \tau_r =$$

$$\frac{y^{(4)}(\ddot{x}+\beta x)-x^{(4)}(\ddot{y}+\beta\dot{y})+\beta(y^{(3)}\ddot{x}-x^{(3)}\ddot{y})-\beta^2(x^{(3)}\dot{y}-y^{(3)}\dot{x})}{[(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2]\cdot}$$
$$-2\frac{[y^{(3)}(\ddot{x}+\beta\dot{x})-x^3(\ddot{y}+\beta\dot{y})-\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})]}{[(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2]^2}\cdot$$
$$\cdot\{(\ddot{x}+\beta\dot{x})(x^{(3)}+\beta\ddot{x})+(\ddot{y}+\beta\dot{y})(y^{(3)}+\beta\ddot{y})\} \quad (18)$$

Thus it is confirmed that the model of the underactuated vessel is a differentially flat one

Example 1: Nonlinear control and state estimation using global linearization

5. Flatness-based control of the underactuated vessel

Next, it will be shown that a flatness-based controller can be developed for the model of the underactuated vessel. It has been shown that it holds

$$\ddot{x} = \dot{u}cos(\psi) - usin(\psi)\dot{\psi} - \dot{v}sin(\psi) - vcos(\psi)\dot{\psi} \Rightarrow \ddot{x} = (vr + \tau_u)cos(\psi) - usin(\psi)r - (-ur - \beta v)sin(\psi) - vcos(\psi)r \Rightarrow \ddot{x} = \tau_u cos(\psi) + \beta v sin(\psi)$$

By differentiating once more with respect to time and after intermediate operations one finally obtains

$$x^{(3)} = \dot{\tau}_u cos(\psi) - \tau_u sin(\psi)r + \\ +\beta(-ur - \beta v)sin(\psi) + \beta v cos(\psi)r \qquad (19)$$

Similarly one has

$$\ddot{y} = \dot{u}sin(\psi) + ucos(\psi)\dot{\psi} + \dot{v}cos(\psi) - vsin(\psi)\dot{\psi} \Rightarrow \ddot{y} = (vr + \tau_u)sin(\psi) + ucos(\psi)r + (-ur - \beta v)cos(\psi) - vsin(\psi)r \Rightarrow \ddot{y} = \tau_u sin(\psi) - \beta v cos(\psi)$$

By differentiating once more with respect to time and by using the state variables of the extended state-space model $z_1 = \tau_u$ and $z_2 = \dot{\tau}_u$ one finally obtains

$$y^{(3)} = z_2 sin(\psi) + z_1 cos(\psi)r + \beta ur cos(\psi) + \\ +\beta^2 v cos(\psi) + \beta v sin(\psi)r$$

Example 1: Nonlinear control and state estimation using global linearization

5. Flatness-based control of the underactuated vessel

Eq. (19) Is differentiated once again with respect to time, so as the control input τ_r to appear

$$x^{(4)} = [-2z_2 sin(\psi)r - z_1 cos(\psi)r^2 - \beta v r^2 sin(\psi) - \beta z_1 r sin(\psi) - \beta u r^2 cos(\psi) + \beta^2 u r sin(\psi) - \beta^3 v sin(\psi) - \beta^2 v r cos(\psi) - \beta u r^2 cos(\psi) + \beta^2 v r cos(\psi) - \beta v r^2 sin\psi] + [cos(\psi)]\ddot{\tau}_u + [-z_1 sin(\psi) - \beta u sin(\psi) + \beta v cos(\psi)]\tau_r \quad (21)$$

Using a Lie algebra-based formulation Eq. (22) Is written in the form

$$x^{(4)} = L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r \quad (22)$$

where

$$L_f^4 y_1 = -2z_2 sin(\psi)r - z_1 cos(\psi)r^2 - \beta v r^2 sin(\psi) - \beta z_1 r sin(\psi) - \beta u r^2 cos(\psi) + \beta^2 u r sin(\psi) - \beta^3 v sin(\psi) - \beta^2 v r cos(\psi) - \beta u r^2 cos(\psi) + \beta^2 v r cos(\psi) - \beta v r^2 sin\psi$$

$$L_{g_a} L_f^3 y_1 = cos(\psi)$$

$$L_{g_b} L_f^3 y_1 = -z_1 sin(\psi) - \beta u sin(\psi) + \beta v cos(\psi)$$

Eq. (20) Is differentiated once again with respect to time, so as the control input $\ddot{\tau}_\theta$ to appear

Example 1: Nonlinear control and state estimation using global linearization

5. Flatness-based control of the underactuated vessel

This gives

$$y^{(4)} = [z_2 r cos(\psi) - z_1 r^2 sin(\psi) + \beta u r^2 sin(\psi) + \beta^2 v r sin(\psi) - \beta v r^2 cos(\psi)] - \beta v r^2 cos(\psi) - \beta z_1 r cos(\psi) + \beta u r^2 sin(\psi) - \beta u r cos(\psi) + \beta^2 v cos(\psi) - \beta^2 v r sin(\psi) + z_2 r cos(\psi)] + [sin(\psi)]\ddot{\tau}_u + [z_1 cos(\psi) - \beta v sin(\psi) - \beta u cos(\psi)]\tau_r$$

which after using a Lie algebra-based formulation is written as

$$y^{(4)} = L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r \quad (23)$$

where $L_f^4 y_2 = [z_2 r cos(\psi) - z_1 r^2 sin(\psi) + \beta u r^2 sin(\psi) - \beta^2 v r sin(\psi) - \beta v r^2 cos(\psi)] - \beta v r^2 cos(\psi) - \beta z_1 r cos(\psi) + \beta u r^2 sin(\psi) - \beta u r cos(\psi) + \beta^2 v cos(\psi) - \beta^2 v r sin(\psi) + z_2 r cos(\psi)]$, and

$$L_{g_a} L_f^3 y_2 = sin(\psi)$$

$$L_{g_b} L_f^3 y_2 = z_1 cos(\psi) - \beta v sin(\psi) - \beta u cos(\psi)$$

Example 1: Nonlinear control and state estimation using global linearization

5. Flatness-based control of the underactuated vessel

Consequently, the aggregate input-output linearized description of the system becomes

$$x^{(4)} = L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r$$
$$y^{(4)} = L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r \quad (24)$$

while by defining the new control inputs

$$v_1 = L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r$$
$$v_2 = L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r \quad (25)$$

one gets

$$x^{(4)} = v_1$$
$$y^{(4)} = v_2 \quad (26)$$

For the dynamics of the linearized equivalent model of the system the following new state variables can be defined

$$z_{1,1} = x \quad z_{1,2} = \dot{x} \quad z_{1,3} = \ddot{x} \quad z_{1,4} = x^{(3)}$$
$$z_{2,1} = y \quad z_{2,2} = \dot{y} \quad z_{2,3} = \ddot{y} \quad z_{2,4} = y^{(3)} \quad (27)$$

Example 1: Nonlinear control and state estimation using global linearization

5. Flatness-based control of the underactuated vessel

and the state-space description of the system becomes

$$\dot{z} = Az + Bv$$
$$z^m = Cz$$

or equivalently

$$\begin{pmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \\ \dot{z}_{1,4} \\ \dot{z}_{2,1} \\ \dot{z}_{2,2} \\ \dot{z}_{2,3} \\ \dot{z}_{2,4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \\ z_{1,4} \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ z_{2,4} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

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while the associated measurement equation is

$$\begin{pmatrix} z_1^m \\ z_2^m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \\ z_{1,4} \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ z_{2,4} \end{pmatrix}$$

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Example 1: Nonlinear control and state estimation using global linearization

5. Flatness-based control of the underactuated vessel

A suitable feedback control law for the linearized system is

$$v_1 = x_d^{(4)} - k_1^1(x^{(3)} - x_d^{(3)}) - k_2^1(\ddot{x} - \ddot{x}_d) - k_3^1(\dot{x} - \dot{x}_d) - k_4^1(x - x_d), \text{ and } v_2 = y_d^{(4)} - k_1^2(y^{(3)} - y_d^{(3)}) - k_2^2(\ddot{y} - \ddot{y}_d) - k_3^2(\dot{y} - \dot{y}_d) - k_4^2(y - y_d) \quad (31)$$

One can compute again the control input that is finally applied to the model of the underactuated vessel. It holds that

$$\bar{v} = \tilde{f} + \tilde{M}\tilde{v} \quad (32)$$

where the following matrices and vectors are defined:

$$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \tilde{f} = \begin{pmatrix} L_f^4 z_{1,1} \\ L_f^4 z_{2,1} \end{pmatrix} \quad (33)$$

$$\tilde{M} = \begin{pmatrix} L_{g,a}L_f^3 z_{1,1} & L_{g_b}L_f^3 z_{1,1} \\ L_{g,a}L_f^3 z_{2,1} & L_{g,b}L_f^3 z_{2,1} \end{pmatrix} \qquad \tilde{v} = \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \end{pmatrix}$$

The stabilizing control input that is finally exerted on the vessel is

$$\tilde{v} = \tilde{M}^{-1}(\bar{v} - \tilde{f}) \quad (34)$$

Example 1: Nonlinear control and state estimation using global linearization

5. Flatness-based control of the underactuated vessel

For the linearized equivalent model of the system it is possible to perform state estimation using the Derivative-free nonlinear Kalman Filter.

Before computing the Kalman Filter stages, the previously defined matrices A,B and C are substituted by their discrete-time equivalents A_d, B_d and C_d.

This is done through common discretization methods. The recursion of the filter's algorithm consists of two stages:

Measurement update::

$$K(k) = P^- C_d^T [P^- C_d^T P + R]^{-1}$$
$$\hat{z}(k) = \hat{z}^-(k) - K(k)[C_d z(k) - C_d \hat{z}^-(k)]$$
$$P(k) = P^-(k) - K(k) C_d P^-(k) \quad (36)$$

Time update::

$$P^-(k+1) = A_d^T P(k) A_d + Q(k)$$
$$\hat{z}^-(k+1) = A_d \hat{z}(k) + B_d u(k) \quad (37)$$

Moreover, using the inverse transformations described by Eq. (3) (7) (9) (10)

one obtains estimates for the state variables of the initial nonlinear system.

Example 1: Nonlinear control and state estimation using global linearization

6. Disturbances compensation with the use of Kalman Filtering

It is assumed that the input-output linearized equivalent model of the system, is subjected to disturbance terms which express the effects of both modelling uncertainty and of external perturbations. Thus one has

$$x^{(4)} = v_1 + \tilde{d}_1$$
$$y^{(4)} = v_2 + \tilde{d}_2 \qquad (38)$$

It is considered that the disturbance signals are equivalently represented by their time derivatives (up to order n) and by the associated initial conditions (however, since disturbances are estimated with the use of the Kalman Filter, finally the dependence on knowledge of initial conditions becomes obsolete). It holds that

$$\tilde{d}_1^{(n)} = f_{d_1} \quad \tilde{d}_2^{(n)} = f_{d_2} \qquad (39)$$

The state vector of the system is extended to include as additional tate variables the disturbance inputs and their derivatives. Thus one obtains

$$z_{d,1} = \tilde{d}_1 \quad z_{d,2} = \dot{\tilde{d}}_1 \quad z_{d,3} = \tilde{d}_2 \quad z_{d,4} = \dot{\tilde{d}}_2 \qquad (40)$$

Thus, the extended state-space description of the system becomes:

Example 1: Nonlinear control and state estimation using global linearization

6. Disturbances compensation with the use of Kalman Filtering

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Thus, the extended state-space description of the system becomes:

Example 1: Nonlinear control and state estimation using global linearization

6. Disturbances compensation with the use of Kalman Filtering

$$\begin{pmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \\ \dot{z}_{1,4} \\ \dot{z}_{2,1} \\ \dot{z}_{2,2} \\ \dot{z}_{2,3} \\ \dot{z}_{2,4} \\ \dot{z}_{d,1} \\ \dot{z}_{d,2} \\ \dot{z}_{d,3} \\ \dot{z}_{d,4} \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \\ z_{1,4} \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ z_{2,4} \\ z_{d,1} \\ z_{d,2} \\ z_{d,3} \\ z_{d,4} \end{pmatrix} + \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \\ f_{d_1} \\ f_{d_2} \end{pmatrix}$$

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42

and the measurement equation becomes

$$\begin{pmatrix} z_{1,1} \\ z_{2,1} \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \end{pmatrix} z_e$$

where $z_e = [z_{1,1}, z_{1,2}, z_{1,3}, z_{1,4}, z_{2,1}, z_{2,2}, z_{2,3}, z_{2,4}, z_{d,1}, z_{d,2}, z_{d,3}, z_{d,4}]^T$

Thus, the extended state-space description of the system becomes:

Example 1: Nonlinear control and state estimation using global linearization

6. Disturbances compensation with the use of Kalman Filtering

$$\dot{z}_e = A_e z_e + B_e v_e$$
$$z_e^{meas} = C_e z_e$$ (43)

For the extended state-space description of the system one can design a state estimator of the form

$$\dot{\hat{z}}_e = A_o z_e + B_o v_e + K(z_e^{meas} - C_o \hat{z}_e)$$
$$\hat{z}_e^{meas} = C_o \hat{z}_e$$ (44)

where for matrices A_o and C_o it holds $A_o = A$ and $C_o = C$ while for matrix B_o it holds

$$B_o^T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Again the Kalman Filter recursion provides joint estimation of the non-measurable state vector elements, of the disturbances' inputs and of their derivatives.

Prior to computing the Kalman Filter stages, the previously defined matrices A,B and C are substituted by their discrete-time equivalents A_{ed} , B_{ed} and C_{ed}.

Example 1: Nonlinear control and state estimation using global linearization

6. Disturbances compensation with the use of Kalman Filtering

The recursion of the filter's algorithm consists of two stages. Thus, one has

Measurement update::

$$K(k) = P_e^- C_{e_d}^T [P_e^- C_{e_d}{}^T P_e + R_e]^{-1}$$
$$\hat{z}_e(k) = \hat{z}_e^-(k) - K(k)[C_{e_d} z_e(k) - C_{e_d} \hat{z}e^-(k)]$$
$$P_e(k) = P_e^-(k) - K(k) C_{e_d} P_e^-(k) \quad (45)$$

Time update::

$$P_e^-(k+1) = A_{e_d}{}^T P_e(k) A_{e_d} + Q_e(k)$$
$$\hat{z}_e^-(k+1) = A_{e_d} \hat{z}_e(k) + B_{e_d} v_e(k) \quad (46)$$

For compensating the disturbances effects, the modified control input that is applied to the system is

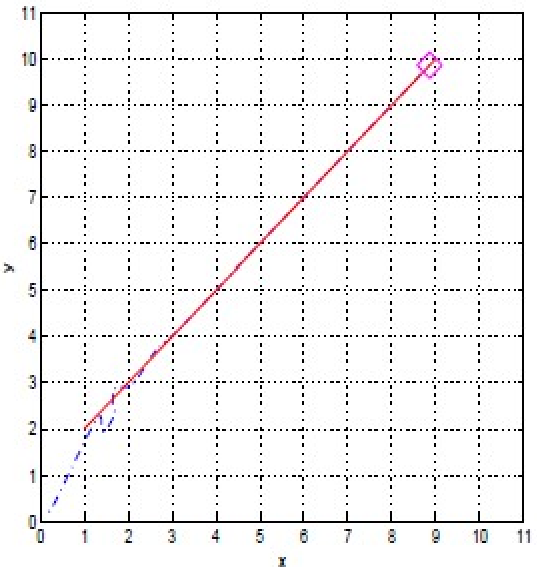
$$v_1 = x_d^{(4)} - k_1^1(x^{(3)} - x_d^{(3)}) - k_2^1(\ddot{x} - \ddot{x}_d) - k_3^1(\dot{x} - \dot{x}_d) - k_4^1(x - x_d) - \hat{z}_{d,1} \text{ and } v_2 = y_d^{(4)} - k_1^2(y^{(3)} - y_d^{(3)}) - k_2^2(\ddot{x} - \ddot{y}_d) - k_3^2(\dot{y} - \dot{y}_d) - k_4^2(y - y_d) - \hat{z}_{d,2}. \quad (47)$$

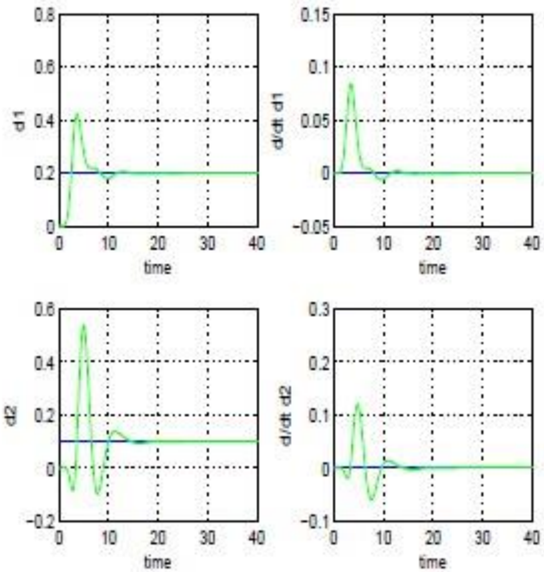
Example 1: Nonlinear control and state estimation using global linearization

7. Simulation tests

In simulation tests It has been observed that in all cases the nonlinear feedback controller succeeded fast and accurate tracking of the reference setpoints.

The Derivative-free nonlinear Kalman Filter enabled estimation of the non-measurable variables of the system's state-vector which were needed for the implementation of the feedback control scheme



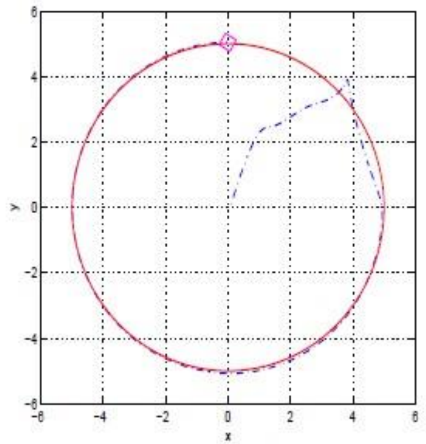


Reference path 1: Trajectory tracking for states x,y of the underactuated hovercraft

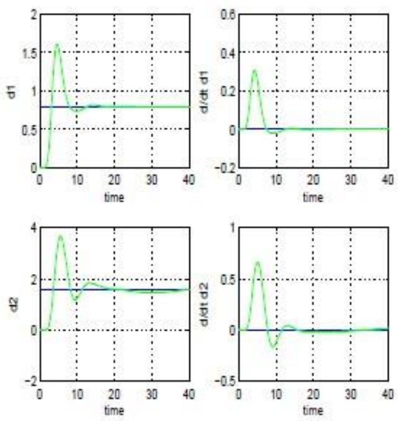
Reference path 1: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter

Example 1: Nonlinear control and state estimation using global linearization

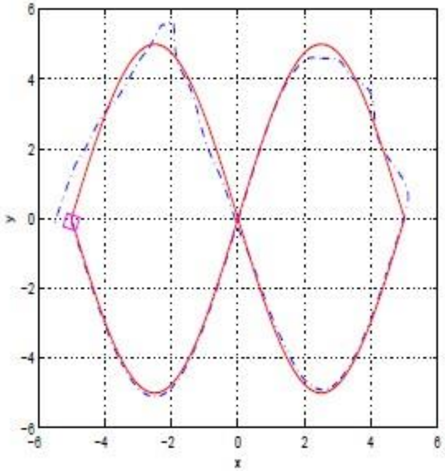
7. Simulation tests



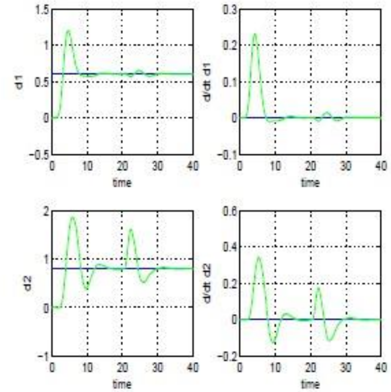
Reference path 2: Trajectory tracking for states x,y of the underactuated hovercraft



Reference path 2: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter



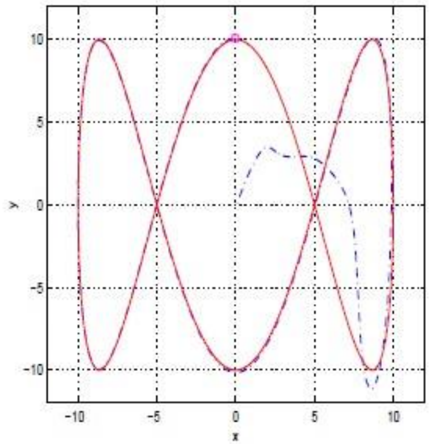
Reference path 3: Trajectory tracking for states x,y of the underactuated hovercraft



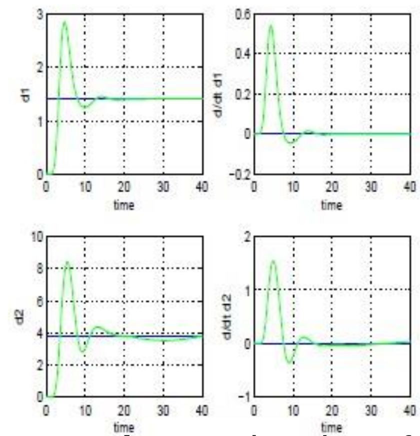
Reference path 3: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter

Example 1: Nonlinear control and state estimation using global linearization

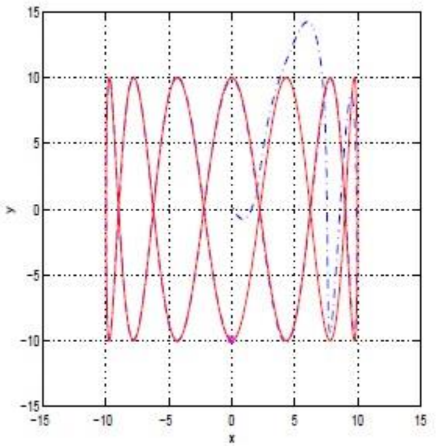
7. Simulation tests



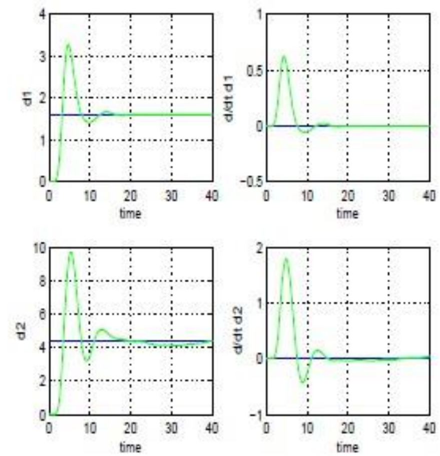
Reference path 4: Trajectory tracking for states x,y of the underactuated hovercraft



Reference path 4: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter



Reference path 5: Trajectory tracking for states x,y of the underactuated hovercraft



Reference path 5: Estimation of disturbance inputs using the Derivative-free non-linear Kalman Filter

Example 1: Nonlinear control and state estimation using global linearization

8. Conclusions

- A **nonlinear control method** has been developed for the **underactuated model** of an **unmanned surface vessel**, based on **differential flatness theory** and on a new nonlinear filtering method under the name **Derivative-free nonlinear Kalman Filter**. First, it was shown that the vessel's model is differentially flat.

- **Dynamic extension** has been used. The system has been augmented by considering as **additional state variables** the control inputs and their derivatives.

- By applying **dynamic extension** and **differential flatness properties**, the vessel's model has been transformed into a **linear form**. Moreover, using the linearized model of the vessel, a **state feedback controller** has been designed.

- Next, to estimate the **non-measurable state variables** of the vessel and to identify additive **disturbance terms** that affected he system, the **Derivative-free nonlinear Kalman Filter** was redesigned as a **disturbance observer.**

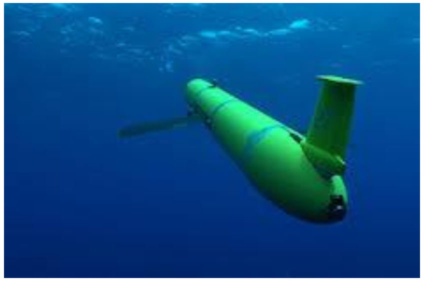
- This algorithm consists of the standard Kalman Filter applied on the linearized equivalent of the system and of an **inverse transformation** that is based on differential flatness theory which computes estimates on the initial nonlinear system.

Example 2: Nonlinear control and state estimation using global linearization

1. Control of a 6-DOF AUV

• A **nonlinear control and filtering** is proposed for Autonomous Underwater Vessels (AUVs) based on **differential flatness theor**y and on the use of the **Derivative-free nonlinear Kalman Filter**.

• First, it is shown that the 6-DOF dynamic model of the AUV is a **differentially flat** one. This enables its transformation into the **linear canonical (Brunovsky) form** and facilitates the design of a state feedback controller.



• A problem that has to be dealt with is the uncertainty about the parameters of the AUV's dynamic model, as well external perturbations which affect its motion.

• To cope with this, it is proposed to use a **disturbance observer** which is based on the **Derivative-free nonlinear Kalman Filter**. This filtering method consists of the standard Kalman Filter recursion applied on the linearized model of the vessel and of an inverse **transformation based on differential flatness theory**, which enables to obtain estimates of the state variables of the initial nonlinear model of the vessel.

• The **Kalman Filter-based disturbance observer** enables the simultaneous estimation of the non-measurable state variables of the AUVs and of the perturbations. By **estimating disturbances**, their compensation is also achieved through suitable modification of the feedback control input.

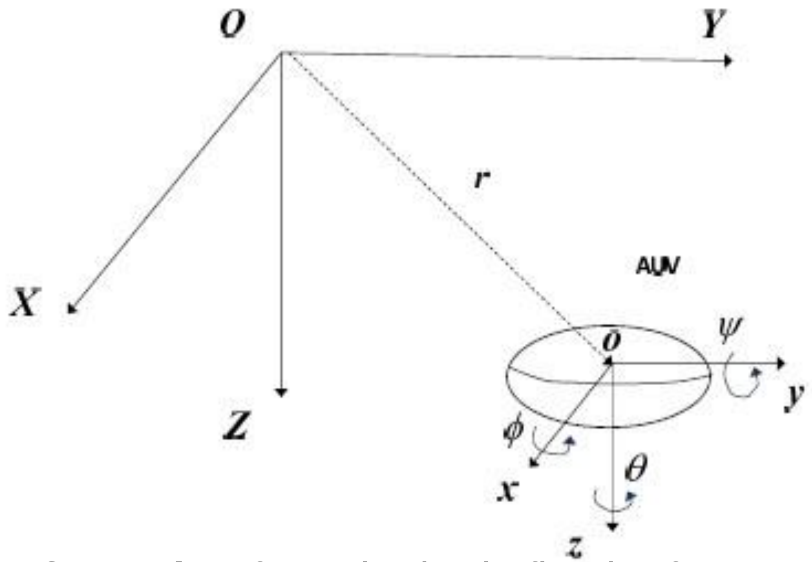
Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Kinematic model of the AUV

The **inertial reference frame** is OXYZ

The **body-fixed reference frame** is O'xyz



The **velocities transformation** from the body-fixed reference frame to the inertial reference frame is given by

$$\dot{\eta}_1 = J_1 v_1 \quad \text{with} \quad \dot{\eta}_1 = [\dot{x}, \dot{y}, \dot{z}]^T \quad v_1 \doteq [u, v, w]^T \qquad (1)$$

$$J_1 = \begin{pmatrix} \cos(\psi)\cos(\theta) & -\sin(\psi)\cos(\phi)+\cos(\psi)\sin(\theta)\sin(\phi) & \sin(\psi)\sin(\phi)+\cos(\psi)\cos(\phi)\sin(\theta) \\ \sin(\psi)\cos(\theta) & \cos(\psi)\cos(\phi)+\sin(\phi)\sin(\theta)\sin(\psi) & -\cos(\psi)\sin(\phi)+\sin(\theta)\sin(\psi)\cos(\phi) \\ -\sin(\theta) & \cos(\theta)\sin(\phi) & \cos(\theta)\cos(\phi) \end{pmatrix}$$

Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Kinematic model of the AUV

Inertial reference frame

State vector

$$x = [x_1, x_2]^T = [x, y, z, \phi, \theta, \psi]^T;$$

Cartesian coordinates

$$x_1 = [x, y, z]^T$$

Euler angles

$$x_2 = [\phi, \theta, \psi]^T$$

Velocities vector

$$\dot{x} = [\dot{x}_1, \dot{x}_2]^T = [\dot{x}, \dot{y}, \dot{z}, \dot{\phi}, \dot{\theta}, \dot{\psi}]^T$$

Linear velocities

$$\dot{x}_1 = [\dot{x}, \dot{y}, \dot{z}]^T$$

Angular velocities

$$\dot{x}_2 = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$$

forces and torques vector

$$\tau = [F_x, F_y, F_z, T_x, T_y, T_z]^T$$

forces

$$\tau_1 = [F_x, F_y, F_z]^T$$

torques

$$\tau_2 = [T_x, T_y, T_z]^T$$

Body-fixed reference frame

Velocities vector

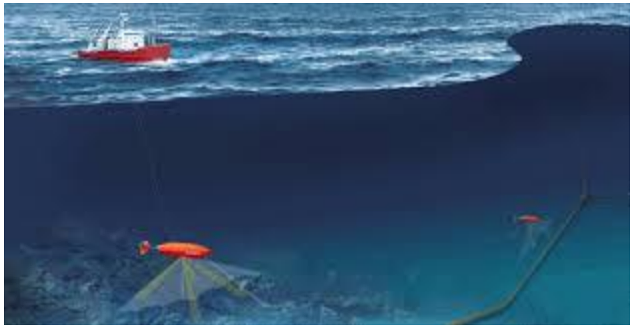
$$v = [v_1, v_2]^T = [u, v, w, p, q, r]^T$$

Linear velocities

$$v_1 = [u, v, w]^T$$

Angular velocities

$$v_2 = [p, q, r]^T$$



Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Kinematic model of the AUV

Moreover, the following **transformation h**olds between **angular velocities expressed in the inertial** and in the body-fixed frame

$$\dot{\eta}_2 = J_2 v_2 \text{ where}$$

$$J_2 = \begin{pmatrix} 1 & sin(\phi)tan(\theta) & cos(\phi)tan(\theta) \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi)/cos(\theta) & cos(\phi)/cos(\theta) \end{pmatrix} \quad (2)$$

Therefore, between the **velocities** in the **body-fixed frame** and in the **inertial reference frame** the following **aggregate transformation** holds

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{or} \quad \dot{\eta} = J \cdot v \quad (3)$$

where $v = [v_1, v_2]^T = [u, v, w, p, q, r]^T$ is the velocities vector n the body-fixed frame

and $\dot{x} = [\dot{x}_1, \dot{x}_2]^T = [\dot{x}, \dot{y}, \dot{z}, \dot{\phi}, \dot{\theta}, \dot{\psi}]^T$ is the velocities vector n the body-fixed frame

Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Dynamic model of the AUV

The **dynamic model of the AUV** representing an equilibrium in forces and torques is

where $$M_{RB}\dot{v} + C_{RB}(v)\cdot v = \tau_{RB} \qquad (4)$$

M_{RB} is the inertia matrix of the AUV.

$C_{RB}(v)$ is the Coriolis and centrifugal forces matrix,

$v = [v_1, v_2]^T = [u, v, w, p, q, r]^T$ is the velocities vector in the body-fixed reference frame

$\tau = [F_x, F_y, F_z, T_x, T_y, T_z]^T$ is the vector of external forces and torques exerted on the AUV

The inertia matrix M_{RB} is given by

$$M_{RB} = \begin{pmatrix} m & 0 & 0 & 0 & mz_G & -my_G \\ 0 & m & 0 & -mz_G & 0 & mx_G \\ 0 & 0 & m & my_G & -mx_G & 0 \\ 0 & -mz_G & my_G & I_x & -I_{xy} & -I_{xz} \\ mz_G & 0 & -mx_G & -I_{xy} & I_y & -I_{yz} \\ -my_G & mx_G & 0 & -I_{xz} & -I_{yz} & I_z \end{pmatrix}$$

Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Dynamic model of the AUV

The **elements of the inertia matrix** of the AUV are:

$I_x,\ I_y,\ I_z$ are inertia matrices

$I_{xy},\ I_{xz},\ I_{yz}$ are inertia products

$r_G = [x_G, y_G, z_G]$ are the coordinates of the AUV's center of mass

The Coriolis matrix of the AUV is given by

$$C_{RB} = \begin{pmatrix} 0 & 0 & 0 & m(y_G q + z_G r) & -m(x_G q - w) & -m(x_G r + v) \\ 0 & 0 & 0 & -m(y_G p + w) & m(z_G r + x_G p) & -m(y_G r - u) \\ 0 & 0 & 0 & -m(z_G p - v) & -m(z_G q + u) & -m(x_G p + y_G q) \\ -m(y_G q + z_G r) & m(y_G p + w) & m(z_G p - v) & 0 & -I_{yz} q - I_{xz} p + I_z r & I_{yz} r + I_{xy} p - I_y q \\ m(x_G q - w) & -m(z_G r + x_G p) & m(z_G q + u) & I_{yz} q + I_{xz} p - I_z r & 0 & I_{xz} r + I_{xy} q + I_x p \\ m(x_G r + v) & m(y_G r - u) & -m(x_G p + y_G q) & -I_{yz} r - I_{xy} p - I_y q & I_{xz} r + I_{xy} q - I_x p & 0 \end{pmatrix}$$

Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Dynamic model of the AUV

The motion of the AUV is also affected by the **inertia of the fluid** that surrounds it:

$$\tau_A = -M_A\dot{v} - C_A(v)v \qquad (5)$$

This means that a **force / torque is developed against the motion** of the vessel and it varies **proportionally to the vessel's acceleration**.

The above inertia matrix M_A is given by

$$M_A = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{pmatrix}$$

and the above Coriolis matrix C_A is given by

$$C_A = \begin{pmatrix} 0 & 0 & 0 & 0 & A_{33}w & -A_{22}v \\ 0 & 0 & 0 & -A_{33}w & 0 & A_{11}u \\ 0 & 0 & 0 & A_{22}v & -A_{11}u & 0 \\ 0 & A_{33}w & -A_{22}v & 0 & A_{66}r & -A_{55}q \\ -A_{33}w & 0 & A_{11}u & -A_{66}r & 0 & A_{44}p \\ A_{22}v & -A_{11}u & 0 & A_{55}q & -A_{44}p & 0 \end{pmatrix}$$

Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Dynamic model of the AUV

The model is completed by the vector of a **force / torque which resists to the motion** of the underwater vessel and which is **proportional to its velocity**

$\tau_{DL} = -D(v)v$ where

$$D(v) = \begin{pmatrix} X_{|u|u}|u| & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_{|v|v}|v| & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{|w|w}|w| & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{|p|p}|p| & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{|q|q}|q| & 0 \\ 0 & 0 & 0 & 0 & 0 & N_{|r|r}|r| \end{pmatrix}$$

while the diagonal elements of matrix D(v) are defined as follows

$$\begin{aligned} X_{|u|u} &= \tfrac{\rho}{2}V^{\frac{2}{3}}C_x(0^\circ,0^\circ) & K_{|p|p} &= \tfrac{\rho}{2}V^{\frac{5}{3}}C_p \\ Y_{|v|v} &= \tfrac{\rho}{2}V^{\frac{2}{3}}C_y(90^\circ,0^\circ) & M_{|q|q} &= \tfrac{\rho}{2}V^{\frac{5}{3}}C_q \\ Z_{|w|w} &= \tfrac{\rho}{2}V^{\frac{2}{3}}C_z(90^\circ,90^\circ) & N_{|r|r} &= \tfrac{\rho}{2}V^{\frac{5}{3}}C_r \end{aligned}$$

where $C_x, C_y, C_z, C_p, C_q, C_r$ are constants

ρ is the specific mass of the water and V is the volume of the submerged vessel

Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Dynamic model of the AUV

There are also **torques and forces** which are exerted on the vessel and which are due to the vessel's weight and lift force. Using that in the inertial reference frame these forces are

$$B = \rho g V \quad \text{and} \quad W = mg:$$

the forces in the body-fixed frame are

$$f_W = J_1^{-1}[0, 0, W]^T \quad \text{and} \quad f_B = -J_1^{-1}[0, 0, B]^T$$

and taking that the associated distance vectors from the origin are

$$r_G = [x_G, y_G, z_G]^T \quad \text{and} \quad r_B = [x_B, y_B, z_B]^T$$

the generated torques are computed in the body-fixed frame are

$$\tau_W = r_G \times f_W \quad \text{and} \quad \tau_B = r_B \times f_B$$

By applying one more transformation on the aforementioned vector, with the use of J_1 the forces and torques due to the effects of weight and lift are finally expressed in the **inertial reference fram**

$$\tau_{WB} = \begin{pmatrix} f_w + f_B \\ \tau_w + \tau_B \end{pmatrix} \Rightarrow$$

$$\tau_{WB} = \left(\tau_w^{1,1} \quad \tau_w^{2,1} \quad \tau_w^{3,1} \quad \tau_w^{4,1} \quad \tau_w^{5,1} \quad \tau_w^{6,1}\right)^T$$

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Example 2: Nonlinear control and state estimation using global linearization

2. Kinematic and dynamic 6-DOF model of the AUV

Dynamic model of the AUV

Thus, due to the effects of **the resistive forces and torques** which are generated by the surrounding fluid one has the dynamics

$$M_{RB}\dot{v} + C_{RB}(v)v = \tau_A + \tau_{DL} + \tau_{WB} + \tau \quad (7)$$

$\tau_A = -M_A\dot{v} - C_A(v)v, \ \tau_{DL} = -D(v)v$ forces/ torques resisting the vessel's motion

$\tau_{WB} = -g_f$ forces/ torques due to weight and lift effects

τ forces/ torques defining the vessel's propulsion

By substituting Eq (5) and Eq (6). Into Eq. (7) one obtains the **aggregate dynamics**

$$(M_{RB} + M_A)\dot{v} + (C_{RB}(v) + C_A(v))v + D(v)v + g_f = \tau \quad (8)$$

where $M = M_{RB} + M_A$; Is the aggregate inertia matrix

$C(v) = C_{RB}(v) + C_A(v)$ Is the aggregate Coriolis matrix

Thus, the **dynamic and the kinematic models of the AUV** are finally written as

$$M\dot{v} + Cv + D(v)v + g_f = \tau \quad (9)$$

$$\dot{\eta} = J(\eta)v \quad (10)$$

Example 2: Nonlinear control and state estimation using global linearization

3. Differential flatness of the 6-DOF model of the AUV

Using that $v = J^{-1}\dot{\eta}$ or $v = R\dot{\eta}$ Eq. (10) can be written as

$$\bar{M}\ddot{\eta} + \bar{C}\dot{\eta} + \bar{D}(\dot{\eta})\dot{\eta} + g_f(\eta) = \tau \qquad (11)$$

where η has been defined in the **inertial reference frame** , while it holds

$$\eta = [x, y, z, \phi, \theta, \psi]^T, \ \bar{M} = MR, \ \bar{C} = M\dot{R} + CR \quad \text{and} \ \bar{D} = DR$$

Moreover, by defining the **inverse matrix** $\bar{M}^{-1} = N$ one obtains

$$\ddot{\eta} + N{\cdot}\bar{C}\dot{\eta} + N{\cdot}\bar{D}(\dot{\eta})\dot{\eta} + N{\cdot}g_f(\eta) = N{\cdot}\tau \qquad (12)$$

Moreover, using the state vector elements notation

$$\begin{aligned}
z_1 &= x, & z_2 &= \dot{x}, & z_3 &= y, & z_4 &= \dot{y}, \\
z_5 &= z, & z_6 &= \dot{z}, & z_7 &= \phi, & z_8 &= \dot{\phi}, \\
z_9 &= \theta, & z_{10} &= \dot{\theta}, & z_{11} &= \psi, & z_{12} &= \dot{\psi}
\end{aligned}$$

the dynamic model of the AUV becomes

Example 2: Nonlinear control and state estimation using global linearization

3. Differential flatness of the 6-DOF model of the AUV

and using that

$f_i(Z)\ i=1,\cdots,6$ are the row elements of vector $f = N{\cdot}\bar{C}\dot{\eta} + N{\cdot}\bar{D}(\dot{\eta})\dot{\eta} + N{\cdot}g_f(\eta)$

$N_i(Z),\ i=1,\cdots,6$: are the rows of matrix $N = M^{-1}$

one obtains

$$\begin{array}{ll} \dot{z}_1 = z_2 & \dot{z}_2 + f_1(Z) = N_1(Z)\tau \\ \dot{z}_3 = z_4 & \dot{z}_4 + f_2(Z) = N_2(Z)\tau \\ \dot{z}_5 = z_6 & \dot{z}_6 + f_3(Z) = N_3(Z)\tau \\ \dot{z}_7 = z_8 & \dot{z}_8 + f_4(Z) = N_4(Z)\tau \\ \dot{z}_9 = z_{10} & \dot{z}_{10} + f_5(Z) = N_5(Z)\tau \\ \dot{z}_{11} = z_{12} & \dot{z}_{12} + f_6(Z) = N_6(Z)\tau \end{array} \quad (13)$$

Next, by denoting the **flat output of the system** as $Y = [z_1, z_3, z_5, z_7, z_9, z_{11}]$.

it holds that $z_2 = \dot{z}_1,\ z_4 = \dot{z}_3,\ z_6 = \dot{z}_5,\ z_8 = \dot{z}_7,\ z_{10} = \dot{z}_9$ and $z_{12} = \dot{z}_{11}$

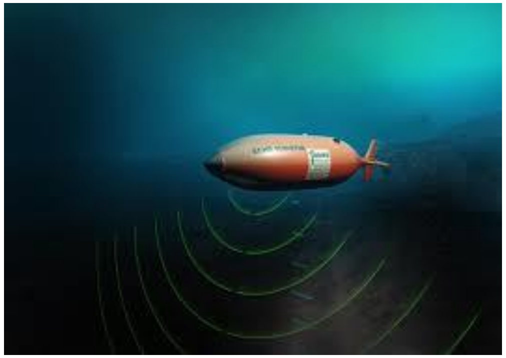
and

$$\begin{array}{ll} z_2 = [1\ 0\ 0\ 0\ 0\ 0]\dot{Y} & z_4 = [0\ 1\ 0\ 0\ 0\ 0]\dot{Y} \\ z_6 = [0\ 0\ 1\ 0\ 0\ 0]\dot{Y} & z_8 = [0\ 0\ 0\ 1\ 0\ 0]\dot{Y} \\ z_{10} = [0\ 0\ 0\ 0\ 1\ 0]\dot{Y} & z_{12} = [0\ 0\ 0\ 0\ 0\ 1]\dot{Y} \end{array}$$

Consequently the **state vector elements** given above can be written **as differential functions** of the flat output Y .

Example 2: Nonlinear control and state estimation using global linearization

3. Differential flatness of the 6-DOF model of the AUV



Moreover, from Eq, (13) one has

$$\ddot{z}_1 = v_1 = -f_1 + N_1\tau \quad \ddot{z}_3 = v_2 = -f_2 + N_2\tau$$
$$\ddot{z}_5 = v_3 = -f_3 + N_3\tau \quad \ddot{z}_7 = v_4 = -f_4 + N_4\tau$$
$$\ddot{z}_9 = v_5 = -f_5 + N_5\tau \quad \ddot{z}_{11} = v_6 = -f_6 + N_6\tau$$

Therefore, one has

$$\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_3 \\ \ddot{z}_5 \\ \ddot{z}_7 \\ \ddot{z}_9 \\ \ddot{z}_{11} \end{pmatrix} = \begin{pmatrix} -f_1 \\ -f_2 \\ -f_3 \\ -f_4 \\ -f_5 \\ -f_6 \end{pmatrix} + \begin{pmatrix} N_1\tau \\ N_2\tau \\ N_3\tau \\ N_4\tau \\ N_5\tau \\ N_6\tau \end{pmatrix} \qquad (14)$$

which is equivalently written as

$$\ddot{z}_a = -f_a(Z) + N\tau \Rightarrow \tau = N^{-1}(\ddot{z}_a + f_a(Z))$$
$$\Rightarrow \tau = M(\ddot{z}_a + f_a(Z)) \qquad (15)$$

Consequently, the **control inputs of the 6-DOF AUV model** can be also written as **functions of the flat output and its derivatives**. Therefore, the **AUV model is a differentially flat one**.

Example 2: Nonlinear control and state estimation using global linearization

4. Flatness-based control of the 6-DOF AUV

By exploiting the previously proven **differential flatness properties of the AUV** it will be shown that a **stabilizing feedback controller** can be designed for the AUV model.

Using Eq. (13) one has

$$\begin{array}{ll} v_1 = -f_1 + N_1\tau & v_2 = -f_2 + N_2\tau \\ v_3 = -f_3 + N_3\tau & v_4 = -f_4 + N_4\tau \\ v_5 = -f_1 + N_1\tau & v_6 = -f_1 + N_1\tau \end{array}$$

or equivalently

$$v = -f_a + N\tau \Rightarrow \tau = N^{-1}(v + f_a) \Rightarrow \tau = M(v + f_a)$$

This means that if the **transformed control inputs v** are computed so as to assure asymptotic tracking of the AUV's reference setpoints, one can also find the **real control inputs** which should be exerted on the AUV for succeeding this objective.

According to the above, the dynamic model of the AUV can be written into the **canonical (Brunovsky) form**

$$\begin{array}{llll} \dot{z}_1 = z_2 & \dot{z}_2 = v_1 & \dot{z}_3 = z_4 & \dot{z}_4 = v_2 \\ \dot{z}_5 = z_6 & \dot{z}_6 = v_3 & \dot{z}_7 = z_8 & \dot{z}_8 = v_4 \\ \dot{z}_9 = z_{10} & \dot{z}_{10} = v_5 & \dot{z}_{11} = z_{12} & \dot{z}_{12} = v_6 \end{array} \quad (16)$$

Example 2: Nonlinear control and state estimation using global linearization

4. Flatness-based control of the 6-DOF AUV

which also takes the **state-space form** $\dot{Z} = AZ + BV$ (17)

or equivalently one has the following **state-space description** for the system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \\ \dot{z}_7 \\ \dot{z}_8 \\ \dot{z}_9 \\ \dot{z}_{10} \\ \dot{z}_{11} \\ \dot{z}_{12} \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \\ z_{11} \\ z_{12} \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}$$

while the **measurement equation** is

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \\ z_{11} \\ z_{12} \end{pmatrix}$$

Example 2: Nonlinear control and state estimation using global linearization

4. Flatness-based control of the 6-DOF AUV

Thus, using differential flatness theory the AUV's model has been written in a **MIMO linear canonical (Brunovsky) form**, which is both **controllable and observabl**e.

After being written in the linear canonical form the AUV's state-space equation comprises subsystems of the form

$$\ddot{y}_{f_i} = v_i, \; i = 1, \cdots, 6 \quad (18)$$

For each one of these subsystems a **controller** can be defined as follows

$$v_i = \ddot{y}_{f_i}^d - k_{d_i}(\dot{y}_{f_i} - \dot{y}_{f_i}^d) - k_{p_i}(y_{f_i} - y_{f_i}^d), \; i = 1, \cdots, 6 \quad (19)$$

The **tracking error dynamics**s becomes $\ddot{e}_i + k_{d_i}\dot{e}_i + k_{p_i}e_i = 0 \quad i = 1,2,\ldots,6$

By selecting the **feedback gains** so as the characteristic polynomials

$$p(s) = s^2 + k_{d_i}s + k_{p_i} \quad i = 1,2,\ldots,6$$

to have poles in the **left complex semiplane** it is assured that

$$\lim_{t\to\infty} e_i(t) = 0 \quad i = 1,2,\ldots,6 \quad (20)$$

Once the transformed control inputs vector $v \in R^{6\times 1}$ has been computed, one can use Eq. (25) to find also the torques and forces vector $\tau = M(v + f_a)$ that should be exerted on the UAV so achieving **setpoints tracking**.

Example 2: Nonlinear control and state estimation using global linearization

5. Disturbances compensation with Derivative-free nonlinear Kalman Filtering

Next, it is assumed that the AUV's model is affected by **additive input disturbances**, thus one has

$$\ddot{z}_1 = v_1 + \bar{d}_1 \quad \ddot{z}_2 = v_2 + \bar{d}_2$$
$$\ddot{z}_3 = v_3 + \bar{d}_3 \quad \ddot{z}_4 = v_4 + \bar{d}_4$$
$$\ddot{z}_5 = v_5 + \bar{d}_5 \quad \ddot{z}_6 = v_6 + \bar{d}_6$$

The system's dynamics can be also written as

$$\dot{z}_1 = z_2, \ \dot{z}_2 = v_1 + \bar{d}_1, \ \dot{z}_3 = z_4, \ \dot{z}_4 = v_2 + \bar{d}_2$$
$$\dot{z}_5 = z_6, \ \dot{z}_6 = v_3 + \bar{d}_3, \ \dot{z}_7 = z_8, \ \dot{z}_8 = v_4 + \bar{d}_4, \qquad (21)$$
$$\dot{z}_9 = z_{10}, \ \dot{z}_{10} = v_5 + \bar{d}_5, \ \dot{z}_{11} = z_{12}, \ \dot{z}_{12} = v_6 + \bar{d}_6$$

Without loss of generality, it is assumed that the **dynamics of the disturbances terms** are described by their **second order derivative**, i.e.

$$\ddot{\bar{d}}_i = f_{d_i}, \ i = 1, \cdots, 6$$

Next, the **extended state vector of the system** is defined so as to include disturbance terms as well. Thus one has the additional state variables

$$z_{13} = \bar{d}_1 \quad z_{14} = \dot{\bar{d}}_1 \quad z_{15} = \ddot{\bar{d}}_1 \quad z_{16} = \bar{d}_2 \quad z_{17} = \dot{\bar{d}}_2 \quad z_{18} = \ddot{\bar{d}}_2$$
$$z_{19} = \bar{d}_3 \quad z_{20} = \dot{\bar{d}}_3 \quad z_{21} = \ddot{\bar{d}}_3 \quad z_{22} = \bar{d}_4 \quad z_{23} = \dot{\bar{d}}_4 \quad z_{24} = \ddot{\bar{d}}_4$$
$$z_{25} = \bar{d}_5 \quad z_{26} = \dot{\bar{d}}_6 \quad z_{27} = \ddot{\bar{d}}_5 \quad z_{28} = \bar{d}_6 \quad z_{29} = \dot{\bar{d}}_6 \quad z_{30} = \ddot{\bar{d}}_6$$

Example 2: Nonlinear control and state estimation using global linearization

5. Disturbances compensation with Derivative-free nonlinear Kalman Filtering

Thus, the **disturbed system** can be described by a **state-space equation of the form**

$$\begin{aligned}\dot{z}_f &= A_f z_f + B_f v \\ z_f^{meas} &= C_f z_f\end{aligned} \quad \text{where } A_f \in R^{30\times 30},\ B_f \in R^{30\times 6} \text{ and } C_f \in R^{6\times 30}, \qquad (22)$$

$$A_f = \begin{pmatrix} 0_{1\times 1} & 1 & 0_{1\times 28} \\ 0_{1\times 12} & 1 & 0_{1\times 17} \\ 0_{1\times 3} & 1 & 0_{1\times 26} \\ 0_{1\times 15} & 1 & 0_{1\times 14} \\ 0_{1\times 5} & 1 & 0_{1\times 24} \\ 0_{1\times 18} & 1 & 0_{1\times 11} \\ 0_{1\times 7} & 1 & 0_{1\times 22} \\ 0_{1\times 21} & 1 & 0_{1\times 8} \\ 0_{1\times 9} & 1 & 0_{1\times 20} \\ 0_{1\times 24} & 1 & 0_{1\times 5} \\ 0_{1\times 11} & 1 & 0_{1\times 18} \\ 0_{1\times 27} & 1 & 0_{1\times 2} \\ 0_{1\times 13} & 1 & 0_{1\times 16} \\ 0_{1\times 14} & 1 & 0_{1\times 15} \\ 0_{1\times 30} & & \\ 0_{1\times 16} & 1 & 0_{1\times 13} \\ 0_{1\times 17} & 1 & 0_{1\times 12} \\ 0_{1\times 30} & & \\ 0_{1\times 19} & 1 & 0_{1\times 10} \\ 0_{1\times 20} & 1 & 0_{1\times 9} \\ 0_{1\times 30} & & \\ 0_{1\times 22} & 1 & 0_{1\times 7} \\ 0_{1\times 23} & 1 & 0_{1\times 6} \\ 0_{1\times 30} & & \\ 0_{1\times 25} & 1 & 0_{1\times 4} \\ 0_{1\times 26} & 1 & 0_{1\times 3} \\ 0_{1\times 30} & & \\ 0_{1\times 28} & 1 & 0_{1\times 1} \\ 0_{1\times 29} & 1 & \\ 0_{1\times 30} & & \end{pmatrix} \quad B_f = \begin{pmatrix} 0_{1\times 6} & & \\ 1 & 0_{1\times 5} & \\ 0_{1\times 1} & 1 & 0_{1\times 4} \\ 0_{1\times 6} & & \\ 0_{1\times 2} & 1 & 0_{1\times 3} \\ 0_{1\times 6} & & \\ 0_{1\times 3} & 1 & 0_{1\times 2} \\ 0_{1\times 6} & & \\ 0_{1\times 4} & 1 & 0_{1\times 1} \\ 0_{1\times 6} & & \\ 0_{1\times 5} & 1 & \\ 0_{18\times 6} & & \end{pmatrix} \quad C_f = \begin{pmatrix} 1 & 0_{1\times 29} & \\ 0_{1\times 2} & 1 & 0_{1\times 27} \\ 0_{1\times 4} & 1 & 0_{1\times 25} \\ 0_{1\times 6} & 1 & 0_{1\times 23} \\ 0_{1\times 8} & 1 & 0_{1\times 21} \\ 0_{1\times 10} & 1 & 0_{1\times 19} \end{pmatrix}$$

Example 2: Nonlinear control and state estimation using global linearization

5. Disturbances compensation with Derivative-free nonlinear Kalman Filtering

The dynamics of the disturbance terms $\bar{d}_i,\ i=1,\cdots,6$ are taken to be unknown in the design of the associated disturbances' estimator.

Defining $\bar{A}_d$, $\bar{B}_d$ and $\bar{C}_d$, the **discrete-time equivalents** of matrices $\bar{A}_f$, $\bar{B}_f$ and $\bar{C}_f$ respectively, one has the following dynamics:

$$\dot{\hat{z}}_f = \bar{A}_f \cdot \hat{z}_f + \bar{B}_f \cdot \bar{v} + K(z_f^{meas} - \bar{C}_f \hat{z}_f) \qquad (23)$$

where $K \in R^{30\times 6}$ Is the state estimator's gain. The associated **Kalman Filter-based** disturbance estimator is given by:

measurement update:

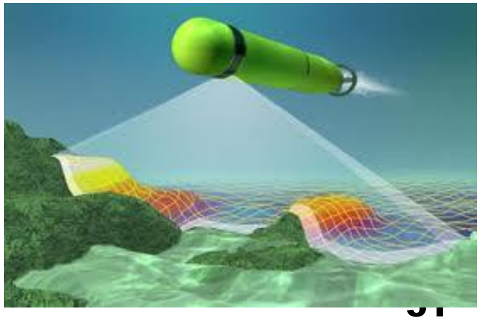
$$K(k) = P^-(k)\bar{C}_d^T[\bar{C}_d P^-(k)\bar{C}_d^T + R]^{-1}$$
$$\hat{z}_f(k) = \hat{z}_f^-(k) + K(k)[z_f^{meas}(k) - \bar{C}_d \hat{z}_f^-(k)]$$
$$P(k) = P^-(k) - K(k)\bar{C}_d P^-(k)$$

time update:

$$P^-(k+1) = \bar{A}_d(k)P(k)\bar{A}_d^T(k) + Q(k)$$
$$\hat{z}_f^-(k+1) = \bar{A}_d(k)\hat{z}_f(k) + \bar{B}_d(k)\bar{v}(k)$$

To **compensate** for the effects of the **disturbance forces** it suffices to use in the control loop the modified control input vector

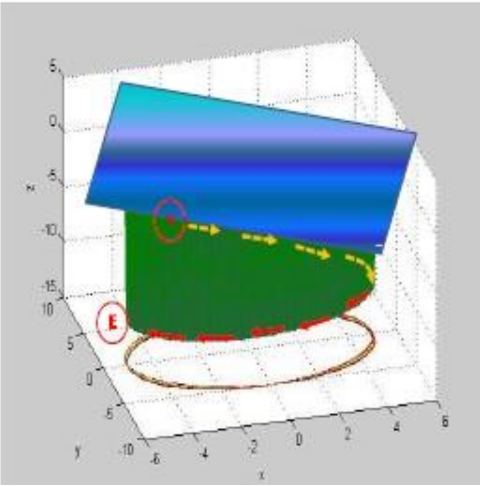
$$v = \begin{pmatrix} v_1 - \hat{d}_1 \\ v_2 - \hat{d}_2 \\ v_3 - \hat{d}_3 \\ v_4 - \hat{d}_4 \\ v_5 - \hat{d}_5 \\ v_6 - \hat{d}_6 \end{pmatrix} \quad \text{or} \quad v = \begin{pmatrix} v_1 - \hat{z}_{13} \\ v_2 - \hat{z}_{16} \\ v_3 - \hat{z}_{19} \\ v_4 - \hat{z}_{22} \\ v_5 - \hat{z}_{25} \\ v_6 - \hat{z}_{28} \end{pmatrix} \qquad (24)$$



Example 2: Nonlinear control and state estimation using global linearization

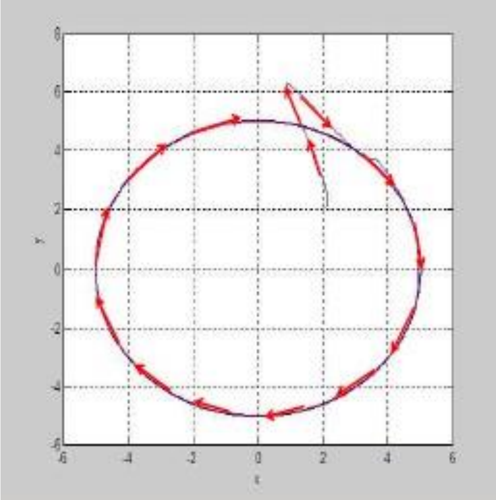
6. Simulation tests

Reference path 1



(a) trajectory of the AUV in the cartesian space

The proposed flatness-based controller enabled fast and accurate tracking of the reference path



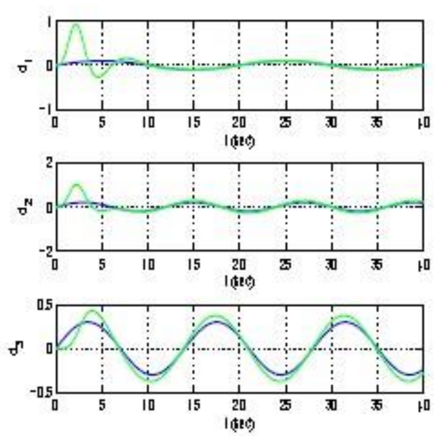
(b) projection of the AUV's trajectory on the xy plane

The Derivative-free nonlinear Kalman Filter, designed as a Disturbance observer enabled estimation and compensation of disturbances

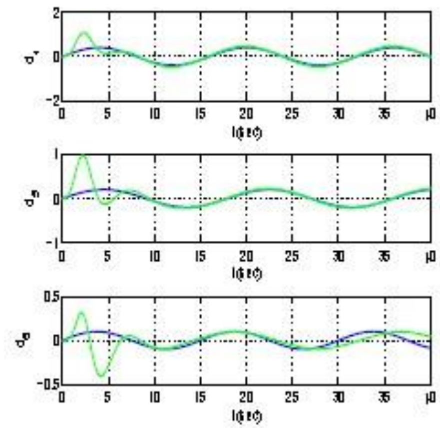
Example 2: Nonlinear control and state estimation using global linearization

6. Simulation tests

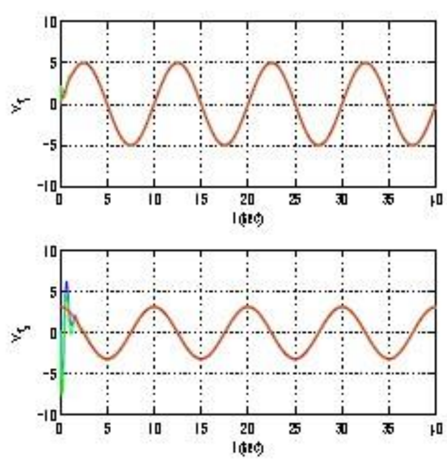
Results about tracking a 3D trajectory, having as projection in the xy-plane a circular path



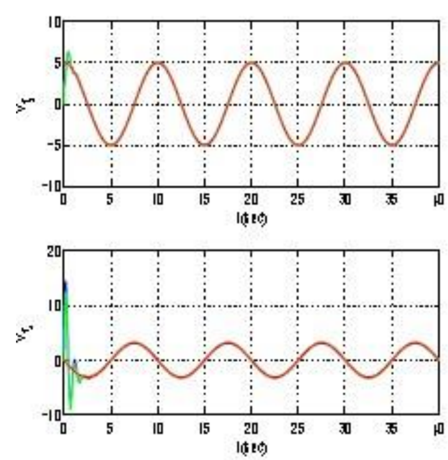
(a) State variables associated with with linear motion of the AUV



(b) State variables associated with with the rotational motion of the AUV



(a) Position and velocity along the x-axis

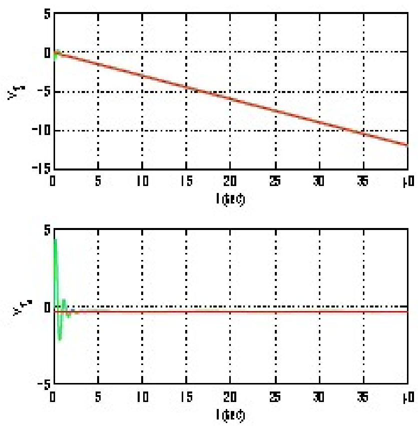


(b) Position and velocity along the y-axis

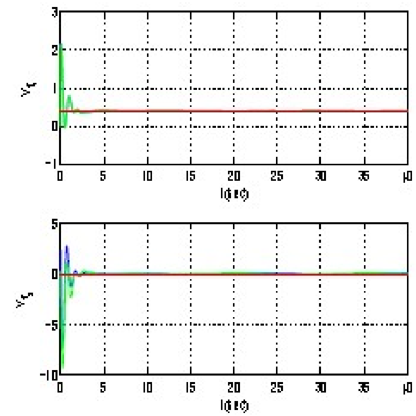
Example 2: Nonlinear control and state estimation using global linearization

6. Simulation tests

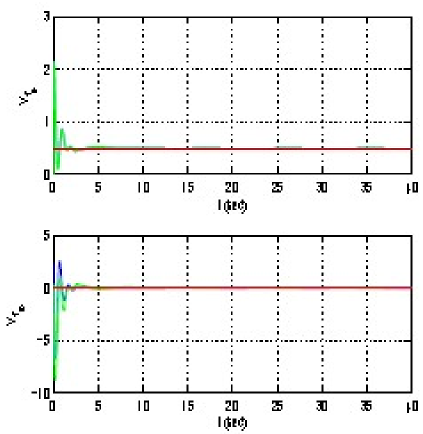
Results about tracking a 3D trajectory, having as projection in the xy-plane a circular path



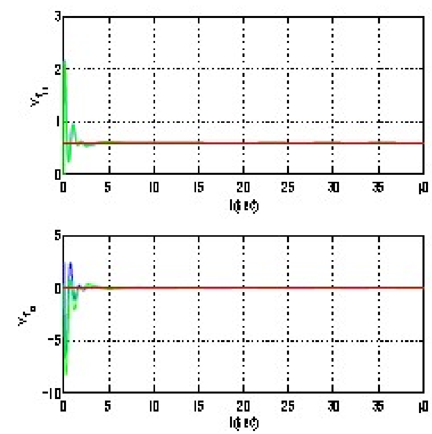
(a) Position and velocity along the z-axis



(b) Rotation angle φ and associated angular speed



(a) Rotation angle θ and associated angular speed

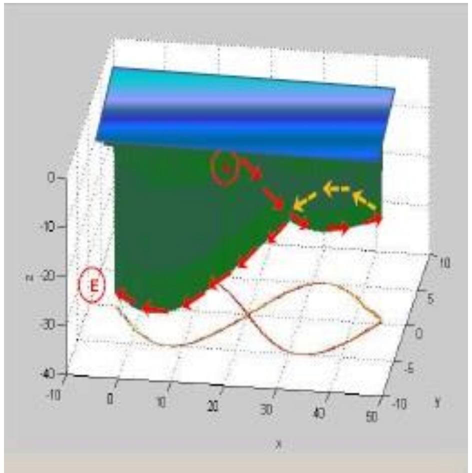


(b) Rotation angle ψ and associated angular speed

Example 2: Nonlinear control and state estimation using global linearization

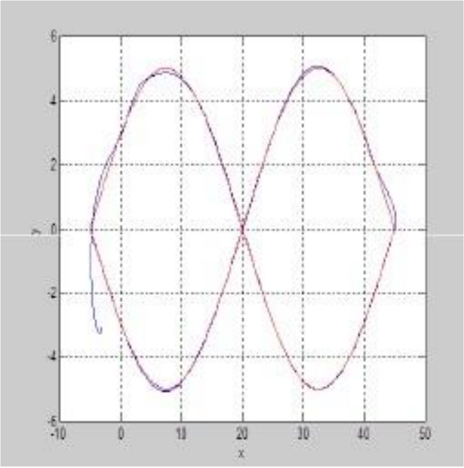
6. Simulation tests

Reference path 2



(a) trajectory of the AUV in the cartesian space

The proposed flatness-based controller enabled fast and accurate tracking of the reference path



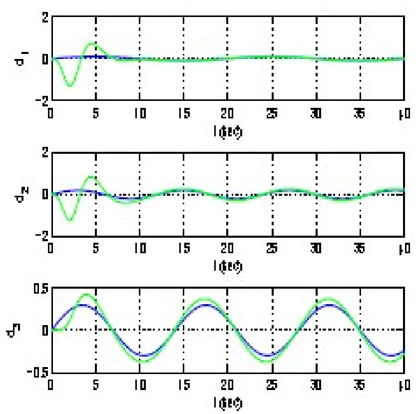
(b) projection of the AUV's trajectory on the xy plane

The Derivative-free nonlinear Kalman Filter, designed as a Disturbance observer enabled estimation and compensation of disturbances

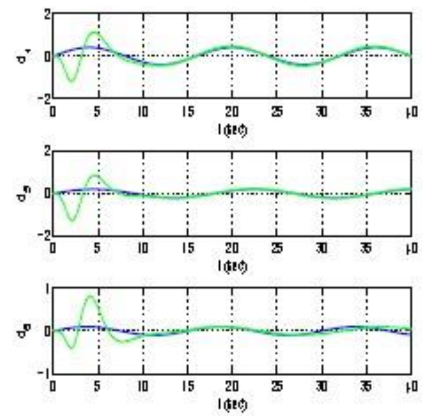
Example 2: Nonlinear control and state estimation using global linearization

6. Simulation tests

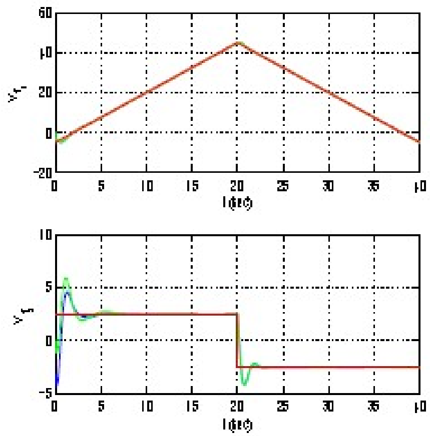
Results about tracking a 3D trajectory, having as projection in the xy-plane an 8-shaped path



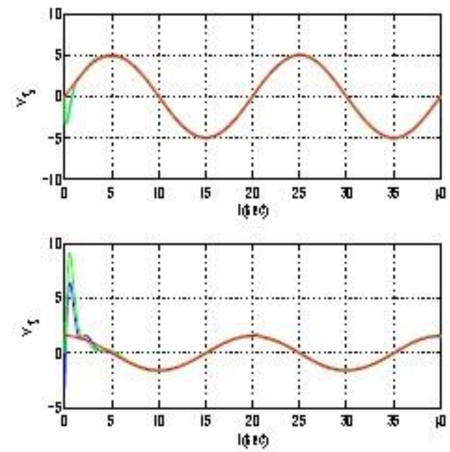
(a) State variables associated with with linear motion of the AUV



(b) State variables associated with with the rotational motion of the AUV



(a) Position and velocity along the x-axis

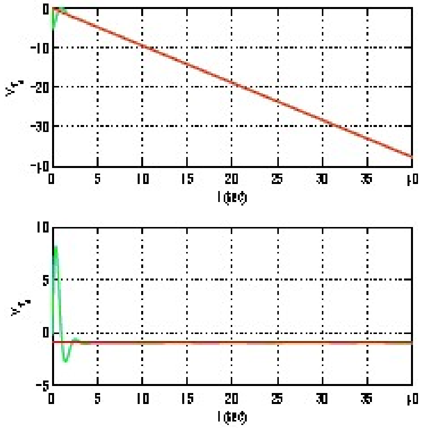


(b) Position and velocity along the y-axis

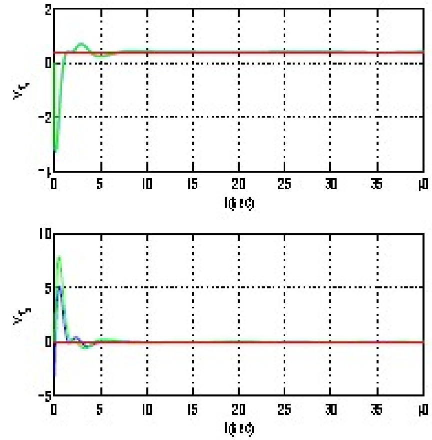
Example 2: Nonlinear control and state estimation using global linearization

6. Simulation tests

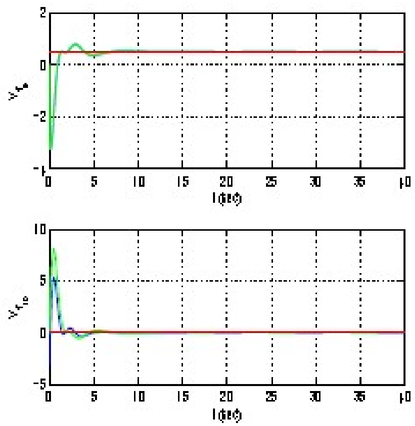
Results about tracking a 3D trajectory, having as projection in the xy-plane an 8-shaped path



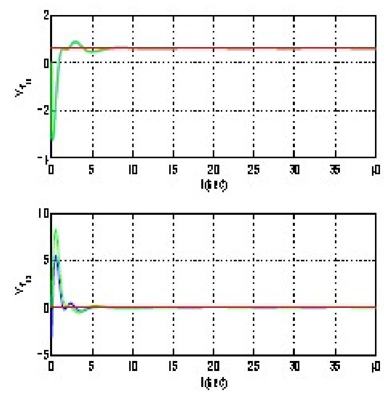
(a) Position and velocity along the z-axis



(b) Rotation angle φ and associated angular speed



(a) Rotation angle θ and associated angular speed



(b) Rotation angle ψ and associated angular speed

Example 2: Nonlinear control and state estimation using global linearization

8. Conclusions

• It was proven that the **dynamic model of the 6-DOF AUV** is a differentially flat one. Next, by exploiting the differential flatness properties of the model, its **transformation into the linear canonical form** has been succeeded.

• For the linearized equivalent description the AUV's dynamics the design of a state feedback controller became possible. Moreover, to **compensate for modelling uncertainties and external perturbations** which affected the AUV's control loop it was proposed to use the **Derivative-free nonlinear Kalman Filter** as a disturbance observer.

• This filter consists of the **Kalman Filter** recursion on the **linearized equivalent model** of the AUV and of an **inverse transformation**, based again on **differential flatness theory**, which enables to obtain estimates of the state variables of the initial nonlinear AUV model.

• By **estimating in real-time the AUV's perturbation input**s, the Derivative-free nonlinear Kalman Filter enabled the compensation of these disturbance terms and the **improvement of the robustness** of the AUV's control loop.

• Finally, the performance of the proposed nonlinear control scheme for AUVs has been confirmed through simulation experiments.

Example 3: Nonlinear control and state estimation using approximate linearization

1. Control of a 3-DOF underactuatd USV

- A **new nonlinear H-infinity control method** is proposed for stabilization and synchronization of **underactuated surface vessels**.

- At first stage **local linearization** of the model of the underactuated vessels is performed round its present operating point.

- The **approximation error** that is introduced to the linearized model is due to **truncation of higher-order terms in the Taylor series expansion** and is represented as a disturbance.

- The control problem is now formulated as **a mini-max differential game** in which the control input tries to minimize the state vector's tracking error while the disturbances affecting the model try to maximize it.

- Using the linearized description of the vessel's dynamics an **H-infinity feedback controller** is designed through the solution of a **Riccati equation** at each step of the control algorithm.

- The **inherent robustness properties of H-infinity control** assure that the disturbance effects will be eliminated and the state variables of the underactuated surface vessel will converge to the desirable setpoints.

- The proposed method, stands for a reliable solution to the problem of **nonlinear control** and stabilization for **unmanned surface vessels** exhibiting underactuation..

Example 3: Nonlinear control and state estimation using approximate linearization

2. Model of the underactuated vessel

- The underactuated vessel's model stems from the generic ship's model, after setting specific values for the elements of the inertia and Coriolis matrix and after reducing the number of the available control inputs.

$$\dot{x} = u\cos(\psi) - v\sin(\psi)$$
$$\dot{y} = u\sin(\psi) + v\cos(\psi)$$
$$\dot{\psi} = r$$
$$\dot{u} = v{\cdot}r + \tau_u$$
$$\dot{v} = -u{\cdot}r - \beta v$$
$$\dot{r} = \tau_r$$

x and *y* are the cartesian coordinates of the vessel

ψ is the orientation angle

u is the surge velocity

v is the sway velocity

r is the yaw rate

The control inputs are the surge force τ_u and the yaw torque τ_r

The underactuated vessel's model is also written in the matrix form

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\psi}\\ \dot{u}\\ \dot{v}\\ \dot{r}\end{pmatrix} = \begin{pmatrix}u\cos(\psi) - v\sin(\psi)\\ u\sin(\psi) + v\cos(\psi)\\ r\\ vr\\ -ur - \beta v\\ 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\tau_u\\ \tau_r\end{pmatrix}$$

Example 3: Nonlinear control and state estimation using approximate linearization

2. Model of the underactuated vessel

or equivalently, one has the description $\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{v}$

The system's state vector is denoted as $\tilde{x} = [x, y, \psi, u, v, r]^T$

while $f(\tilde{x}) \in R^{6\times 1}$ and $\tilde{g}(\tilde{x}) = [\tilde{g}_a, \tilde{g}_b] \in R^{6\times 2}$

while the control input is the vector $\tilde{v} = [\tau_u, \tau_r]^T$

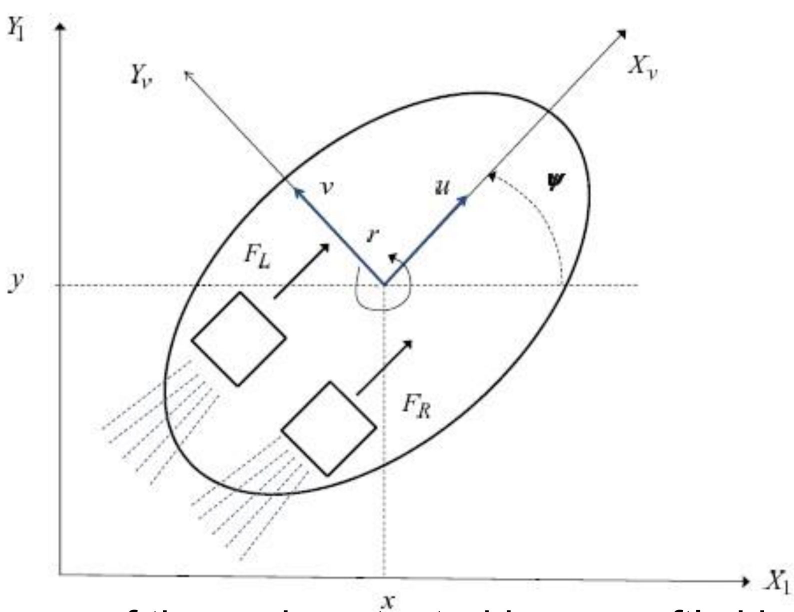


Fig. 1. Diagram of the underactuated hovercraft's kinematic model

Example 3: Nonlinear control and state estimation using approximate linearization

2. Model of the underactuated vessel

The system's state vector can be extended by including as additional state variables the control input τ_u and its first derivative $\dot{\tau}_u$.

The extended state-space description of the system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} ucos(\psi) - vsin(\psi) \\ usin(\psi) + vcos(\psi) \\ r \\ vr + z_1 \\ -ur - \beta v \\ 0 \\ z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \end{pmatrix}$$

or equivalently, one has the description $\dot{z} = f(z) + g(z)\tilde{v}$

The extended system's state vector is denoted as $z = [x, y, \psi, u, v, r, z_1, z_2]^T$.

Moreover, one has $f(z) \in R^{8\times 1}$ and $g(z) = [g_a, g_b] \in R^{8\times 2}$,

while the control input is the vector is $\tilde{v} = [\ddot{\tau}_u, \tau_r]^T$.

Example 3: Nonlinear control and state estimation using approximate linearization

3. Linearization of the model of the underactuated vessel

Local linearization is performed for the state-space model of the underactuated vessel, round the operating point (x^*, u^*) where x^* is the present value of the system's state vector and u^* is the last sampled value of the control inputs vector.

The joint kinematics and dynamics model is written in the form: $\dot{x}=f(x)+g(x)u$

where the state vector is: $x=[x_1,x_2,x_3,x_4,x_5,x_6]^T=[x,y,\psi,u,v,r]^T$; and

$$f(x)=\begin{pmatrix} v\cos(\psi)-v\sin(\psi) \\ u\sin(\psi)+v\cos(\psi) \\ r \\ v\cdot r \\ -ur-\beta v \\ 0 \end{pmatrix} \quad g(x)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and using the state variables notation one gets the description

$$f(x)=\begin{pmatrix} x_4\cos(x_3)-x_5\sin(x_3) \\ x_4\sin(x_3)+x_5\cos(x_3) \\ x_6 \\ x_5x_6 \\ -x_4x_6-\beta x_5 \\ 0 \end{pmatrix} \quad g(x)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 3: Nonlinear control and state estimation using approximate linearization

3. Linearization of the model of the underactuated vessel

The linearization of the vessel's model around the temporary equilibrium gives

$$\dot{x} = Ax + Bu$$

where

$$A = \nabla_x[f(x) + g(x)u] \mid_{(x^*,u^*)} \Rightarrow A = \nabla_x f(x) \mid_{(x^*,u^*)}$$

$$B = \nabla_u[f(x) + g(x)u] \mid_{(x^*,u^*)} \Rightarrow B = g(x) \mid_{(x^*,u^*)}$$

For the Jacobian matrix $A = \nabla_x[f(x) + g(x)u] \mid_{(x^*,u^*)}$ =

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_6} \\ \cdots & \cdots & \cdots & \cdots & \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \frac{\partial f_6}{\partial x_3} & \cdots & \frac{\partial f_6}{\partial x_6} \end{pmatrix}$$

For the first row of the aforementioned Jacobian matrix one has:

$$\frac{\partial f_1}{\partial x_1} = 0, \ \frac{\partial f_1}{\partial x_2} = 0, \ \frac{\partial f_1}{\partial x_3} = -x_4 sin(x_3) - x_5 cos(x_3), \ \frac{\partial f_1}{\partial x_4} = cos(x_3), \ \frac{\partial f_1}{\partial x_5} = -sin(x_3), \ \frac{\partial f_1}{\partial x_6} = 0.$$

Example 3: Nonlinear control and state estimation using approximate linearization

3. Linearization of the model of the underactuated vessel

For the second row of the aforementioned Jacobian matrix one has:

$$\frac{\partial f_2}{\partial x_1}=0,\ \frac{\partial f_2}{\partial x_2}=0,\ \frac{\partial f_2}{\partial x_3}=x_4cos(x_3)-x_5sin(x_3),\ \frac{\partial f_2}{\partial x_4}=sin(x_3),\ \frac{\partial f_2}{\partial x_5}=cos(x_3),\ \frac{\partial f_2}{\partial x_6}=0.$$

For the third row of the aforementioned Jacobian matrix one has:

$$\frac{\partial f_3}{\partial x_1}=0,\ \frac{\partial f_3}{\partial x_2}=0,\ \frac{\partial f_3}{\partial x_3}=0,\ \frac{\partial f_3}{\partial x_4}=0,\ \frac{\partial f_3}{\partial x_5}=0,\ \frac{\partial f_3}{\partial x_6}=1.$$

For the fourth row of the aforementioned Jacobian matrix one has:

$$\frac{\partial f_4}{\partial x_1}=0,\ \frac{\partial f_4}{\partial x_2}=0,\ \frac{\partial f_4}{\partial x_3}=0,\ \frac{\partial f_4}{\partial x_4}=0,\ \frac{\partial f_4}{\partial x_5}=x_6,\ \frac{\partial f_4}{\partial x_6}=x_5.$$

For the fifth row of the aforementioned Jacobian matrix one has:

$$\frac{\partial f_5}{\partial x_1}=0,\ \frac{\partial f_5}{\partial x_2}=0,\ \frac{\partial f_5}{\partial x_3}=0,\ \frac{\partial f_5}{\partial x_4}=-x_6,\ \frac{\partial f_5}{\partial x_5}=-\beta,\ \frac{\partial f_5}{\partial x_6}=-x_6.$$

For the sixth row of the aforementioned Jacobian matrix one has:

$$\frac{\partial f_6}{\partial x_1}=0,\ \frac{\partial f_6}{\partial x_2}=0,\ \frac{\partial f_6}{\partial x_3}=0,\ \frac{\partial f_6}{\partial x_4}=0,\ \frac{\partial f_6}{\partial x_5}=0,\ \frac{\partial f_6}{\partial x_6}=0.$$

Example 3: Nonlinear control and state estimation using approximate linearization

3. Linearization of the model of the underactuated vessel

Parameter d_1 stands for the **linearization error** in the underactuated vessels' model

$$\dot{x} = Ax + Bu + d_1 \qquad (A)$$

The **desirable trajectory** of the underactuated vessel is denoted by

$$x_d = [x_{d_1}, x_{d_2}, x_{d_3}, \ldots, x_{d_7}, x_{d_8}, x_{d_9}]^T$$

Tracking of this trajectory is achieved after applying the control input u^*

At every time instant the control input u^* is assumed to differ from the control input u appearing in (A) by an amount equal to Δu, that is $u^* = u + \Delta u$

$$\dot{x}_d = Ax_d + Bu^* + d_2 \qquad (B)$$

The dynamics of the system of Eq. (A) can be also written in the form

$$\dot{x} = Ax + Bu + Bu^* - Bu^* + d_1 \qquad (C)$$

and by denoting $d_3 = -Bu^* + d_1$ as an **aggregate disturbance** term one obtains

$$\dot{x} = Ax + Bu + Bu^* + d_3 \qquad (D)$$

Example 3: Nonlinear control and state estimation using approximate linearization

4. The nonlinear H-infinity control

where matrices A and B are obtained from the **computation of the Jacobians**

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} |_{(x^*, u^*)} \qquad B = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{pmatrix} |_{(x^*, u^*)}$$

and vector d denotes disturbance terms due to linearization errors.

The problem of **disturbance rejection** for the linearized model that is described by

$$\dot{x} = Ax + Bu + Ld$$
$$y = Cx$$

where $x \in R^n$, $u \in R^m$, $d \in R^q$ and $y \in R^p$ cannot be handled efficiently if the classical LQR control scheme is applied. This because of the existence of the perturbation term d.

In the $H\infty$ control approach, a **feedback control scheme** is designed for **trajectory tracking** by the system's state vector and simultaneous disturbance rejection, considering that the disturbance affects the system in the worst possible manner

Example 3: Nonlinear control and state estimation using approximate linearization

4. The nonlinear H-infinity control

The disturbances' effects are incorporated in the following **quadratic cost function**

$$J(t)=\frac{1}{2}\int_0^T [y^T(t)y(t)+ +ru^T(t)u(t)-\rho^2 d^T(t)d(t)]dt, \quad r,\rho>0$$

The coefficient r determines **the penalization of the control input** and the weight coefficient ρ determines the **reward of the disturbances**' effects. It is assumed that

Then, the **optimal feedback control** law is given by

$$u(t)=-Kx(t) \qquad \text{with} \qquad K=\frac{1}{r}B^TP$$

where P **is a positive semi-definite symmetric matrix** which is obtained from the solution of the **Riccati equation**

$$A^TP+PA+Q-P(\frac{1}{r}BB^T-\frac{1}{2\rho^2}LL^T)P=0$$

where Q is also a positive definite symmetric matrix.

Parameter ρ in Eq. (15), is an **indication of the closed-loop system robustness**. If the values of ρ> 0 are excessively decreased with respect to r, then the solution of the Riccati equation is no longer a positive definite matrix. Consequently, there is a lower bound ρ_{min} of for which the H-infinity control problem has a solution.

Example 3: Nonlinear control and state estimation using approximate linearization

5. Lyapunov stability analysis

The **tracking error dynamics** for the unmanned surface vessel is written in the form

$$\dot{e} = Ae + Bu + L\tilde{d}$$

where in the underactuated vessel's application example $L = I \in R^8$ with I being the identity matrix. The following **Lyapunov function** is considered

$$V = \frac{1}{2}e^T P e$$

where $e = x - x_d$ Is the state vector's tracking error

$$\dot{V} = \frac{1}{2}\dot{e}^T P e + \frac{1}{2}e P \dot{e} \Rightarrow$$

$$\dot{V} = \frac{1}{2}[Ae + Bu + L\tilde{d}]^T P + \frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow$$

$$\dot{V} = \frac{1}{2}[e^T A^T + u^T B^T + \tilde{d}^T L^T]Pe + \\ + \frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow$$

$$\dot{V} = \frac{1}{2}e^T A^T Pe + \frac{1}{2}u^T B^T Pe + \frac{1}{2}\tilde{d}^T L^T Pe + \\ \frac{1}{2}e^T PAe + \frac{1}{2}e^T PBu + \frac{1}{2}e^T PL\tilde{d}$$

Example 3: Nonlinear control and state estimation using approximate linearization

5. Lyapunov stability analysis

The previous equation is rewritten as

$$\dot{V}=\frac{1}{2}e^T(A^TP+PA)e+(\frac{1}{2}u^TB^TPe+\frac{1}{2}e^TPBu)+ \\ +(\frac{1}{2}\tilde{d}^TL^TPe+\frac{1}{2}e^TPL\tilde{d})$$

Assumption: For given positive definite matrix Q *and coefficients* r and ρ there exists a positive definite matrix P, which is the solution of the following matrix equation

$$A^TP+PA=-Q+P(\frac{2}{r}BB^T-\frac{1}{\rho^2}LL^T)P \qquad (G)$$

Moreover, the following **feedback control law** is applied to the system

$$u=-\frac{1}{r}B^TPe \qquad (H)$$

By substituting Eq. (H) and Eq. (G) one obtains

$$\dot{V}=\frac{1}{2}e^T[-Q+P(\frac{2}{r}BB^T-\frac{1}{\rho^2}LL^T)P]e+ \\ +e^TPB(-\frac{1}{r}B^TPe)+e^TPL\tilde{d}\Rightarrow$$

Example 3: Nonlinear control and state estimation using approximate linearization

5. Lyapunov stability analysis

Continuing with computations one obtains

$$\dot{V} = -\frac{1}{2}e^TQe + \frac{1}{r}e^TPBB^TPe - \frac{1}{2\rho^2}e^TPLL^TPe$$
$$-\frac{1}{r}e^TPBB^TPe + e^TPL\tilde{d}$$

which next gives

$$\dot{V} = -\frac{1}{2}e^TQe - \frac{1}{2\rho^2}e^TPLL^TPe + e^TPL\tilde{d}$$

or equivalently

$$\dot{V} = -\frac{1}{2}e^TQe - \frac{1}{2\rho^2}e^TPLL^TPe + \frac{1}{2}e^TPL\tilde{d} + \frac{1}{2}\tilde{d}^TL^TPe \qquad (\text{I})$$

Lemma: The following inequality holds

$$\frac{1}{2}e^TL\tilde{d} + \frac{1}{2}\tilde{d}L^TPe - \frac{1}{2\rho^2}e^TPLL^TPe \leq \frac{1}{2}\rho^2\tilde{d}^T\tilde{d}$$

Example 3: Nonlinear control and state estimation using approximate linearization

5. Lyapunov stability analysis

Proof : *The binomial* $(\rho a - \frac{1}{\rho}b)^2$ is considered. Expanding the left part of the above inequality one gets

$$\rho^2 a^2 + \frac{1}{\rho^2}b^2 - 2ab \geq 0 \Rightarrow \frac{1}{2}\rho^2 a^2 + \frac{1}{2\rho^2}b^2 - ab \geq 0 \Rightarrow$$
$$ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2 a^2 \Rightarrow \frac{1}{2}ab + \frac{1}{2}ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2 a^2$$

The following substitutions are carried out: $a = \tilde{d}$ and $b = e^T P L$ and the previous relation becomes

$$\frac{1}{2}\tilde{d}^T L^T P e + \frac{1}{2}e^T P L \tilde{d} - \frac{1}{2\rho^2}e^T P L L^T P e \leq \frac{1}{2}\rho^2 \tilde{d}^T \tilde{d} \quad \text{(J)}$$

Eq. (J) is substituted in Eq. (I) and the inequality is enforced, thus giving

$$\dot{V} \leq -\frac{1}{2}e^T Q e + \frac{1}{2}\rho^2 \tilde{d}^T \tilde{d} \quad \text{(K)}$$

Eq. (K) shows that the **H-infinity tracking performance criterion** is satisfied.

The integration of $\dot{V}$ from 0 to T gives

$$\int_0^T \dot{V}(t)dt \leq -\frac{1}{2}\int_0^T ||e||_Q^2 dt + \frac{1}{2}\rho^2 \int_0^T ||\tilde{d}||^2 dt \Rightarrow$$
$$2V(T) + \int_0^T ||e||_Q^2 dt \leq 2V(0) + \rho^2 \int_0^T ||\tilde{d}||^2 dt$$

Example 3: Nonlinear control and state estimation using approximate linearization

5. Lyapunov stability analysis

Moreover, if there exists a positive constant $M_d > 0$ such that

$$\int_0^{\infty} ||\bar{d}||^2 dt \leq M_d$$

then one gets

$$\int_0^{\infty} ||e||_Q^2 dt \leq 2V(0) + \rho^2 M_d$$

Thus, the integral $\int_0^{\infty} ||e||_Q^2 dt$ is bounded.

Moreover, V(T) is bounded and from the definition of the Lyapunov function V it becomes clear that **e(t) will be also bounded** since

$$e(t) \in \Omega_e = \{e | e^T P e \leq 2V(0) + \rho^2 M_d\}.$$

According to the above and with the use of **Barbalat's Lemma** one obtains:

$$lim_{t \to \infty} e(t) = 0.$$

This completes the stability proof.

Example 3: Nonlinear control and state estimation using approximate linearization

7. Robust state estimation with the use of the H-infinity Kalman Filter

A discrete-time description of the linearized state-space model of the vessel is assumed. The **H-infinity Kalman Filter**, for the **model of the underactuated vessel**, can be formulated in terms of a **measurement update** and a **time update part**

Measurement update:

$$D(k) = [I - \theta W(k)P^-(k) + C^T(k)R(k)^{-1}C(k)P^-(k)]^{-1}$$
$$K(k) = P^-(k)D(k)C^T(k)R(k)^{-1}$$
$$\hat{x}(k) = \hat{x}^-(k) + K(k)[y(k) - C\hat{x}^-(k)]$$

Time update:

$$\hat{x}^-(k+1) = A(k)x(k) + B(k)u(k)$$
$$P^-(k+1) = A(k)P^-(k)D(k)A^T(k) + Q(k)$$

where θ is sufficiently small to assure positive definiteness for the covariance matrix

$$P^-(k) - \theta W(k) + C^T(k)R(k)^{-1}C(k)$$

One can measure only a subset of the state variables of the vessel's model (e.g. cartesian coordinates) and can **estimate through filtering** the rest of the state vector elements.

Besides the filter can be used for **sensor fusion** purposes.

Example 3: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

• The nonlinear **H-nfinity control scheme** is tested through simulation examples

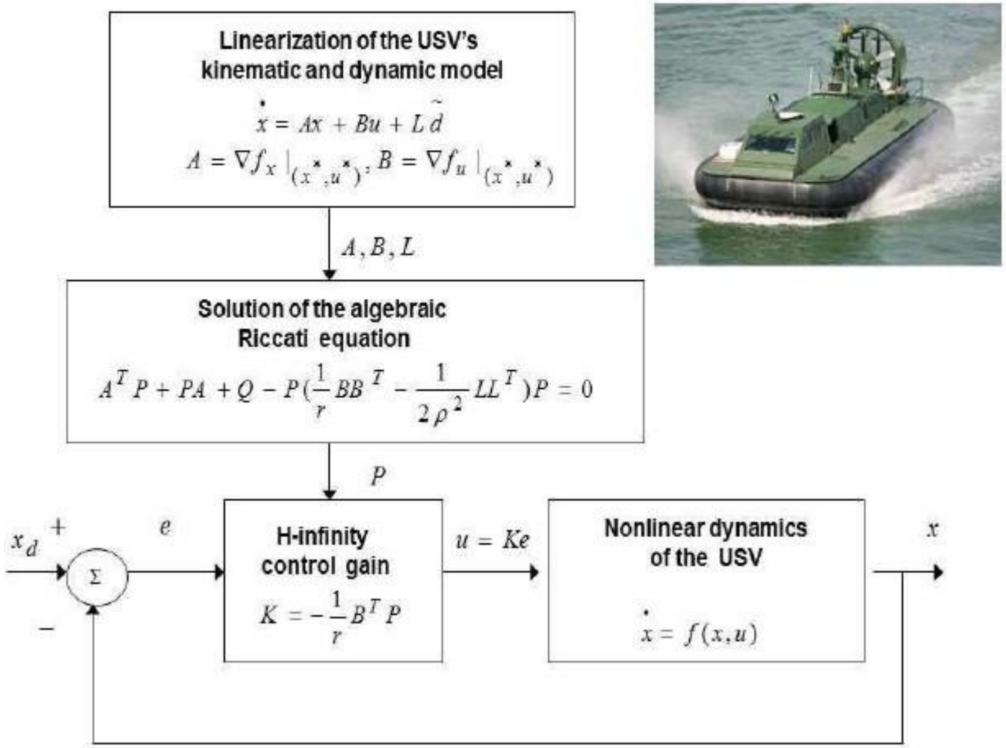


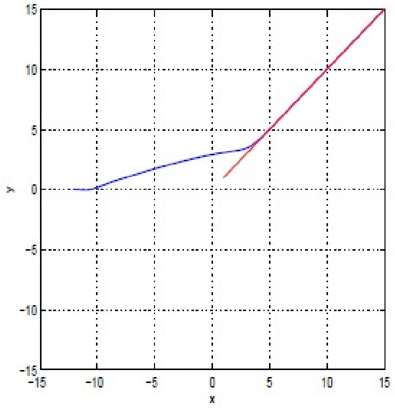
Fig. 2: Diagram of the control scheme for the underactuated vessel

It can be noted that the H-infinity algorithm exhibited remarkable robustness to uncertainty in the model of the distributed power generators which was to approximate linearization.

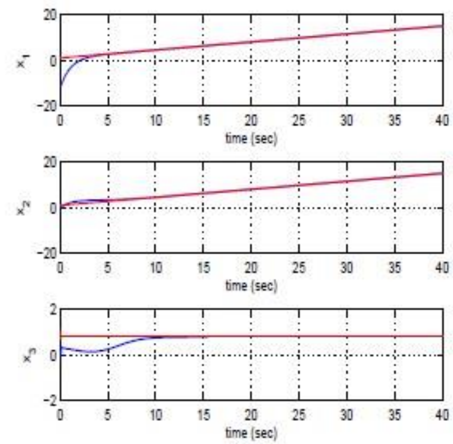
Example 3: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

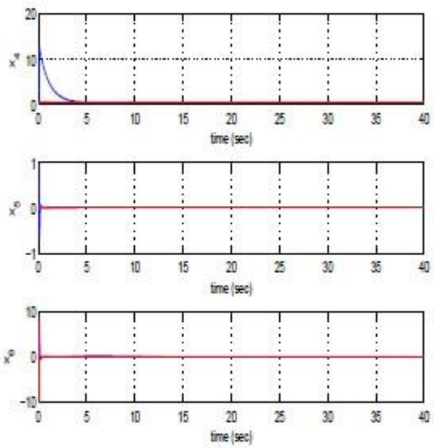
Path 1



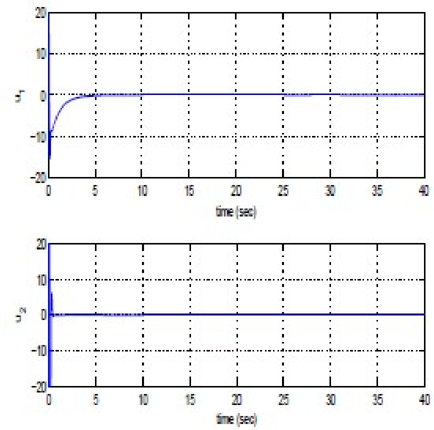
Tracking of the reference trajectory (red line) in the x − y plane by the unmanned surface vessel (blue line),



Convergence of the state variables x1 = x, x2 = y and x3 = ψ to the reference values



Convergence of the state variables of the vessel x4 = u, x5 = v and x6 = r to the reference values

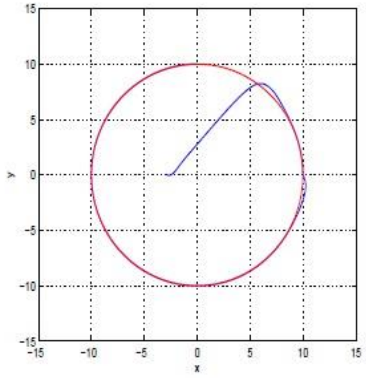


Control inputs u1 and u2 exerted on vessel

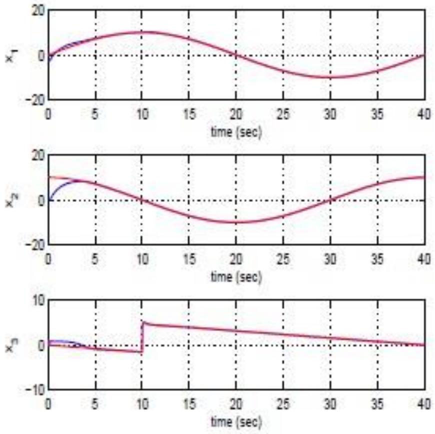
Example 3: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

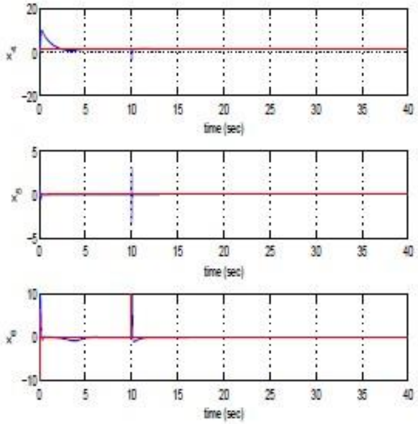
Path 2



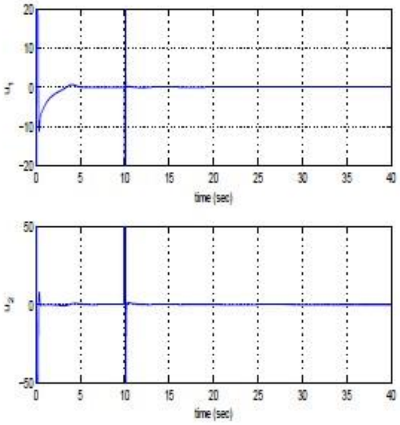
Tracking of the reference trajectory (red line) in the x − y plane by the unmanned surface vessel (blue line),



Convergence of the state variables x1 = x, x2 = y and x3 = ψ to the reference values



Convergence of the state variables of the vessel x4 = u, x5 = v and x6 = r to the reference values

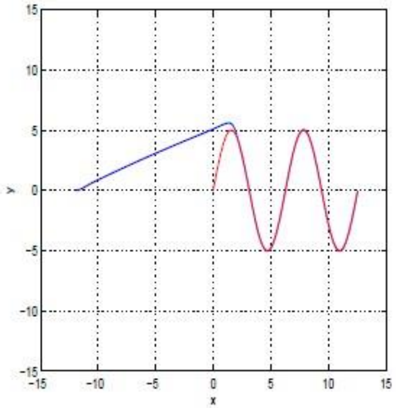


Control inputs u1 and u2 exerted on vessel

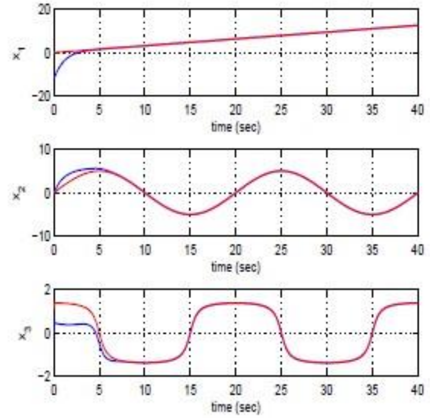
Example 3: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

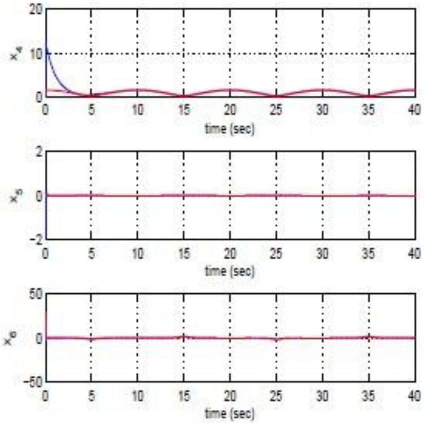
Path 3



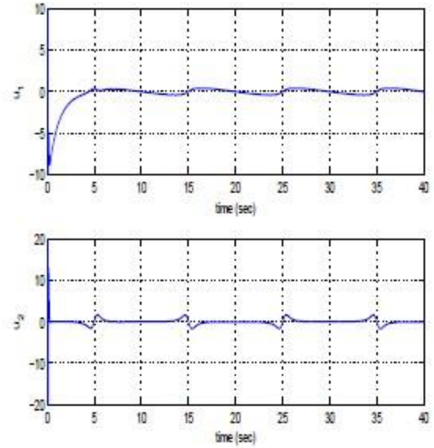
Tracking of the reference trajectory (red line) in the x − y plane by the unmanned surface vessel (blue line),



Convergence of the state variables x1 = x, x2 = y and x3 = ψ to the reference values



Convergence of the state variables of the vessel x4 = u, x5 = v and x6 = r to the reference values

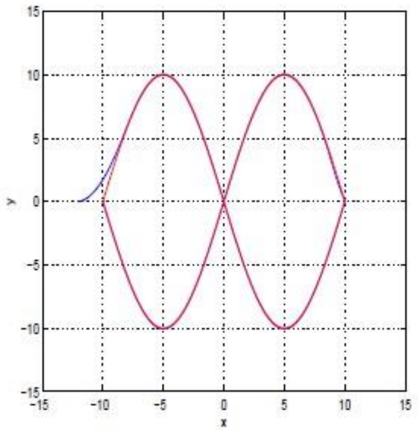


Control inputs u1 and u2 exerted on vessel

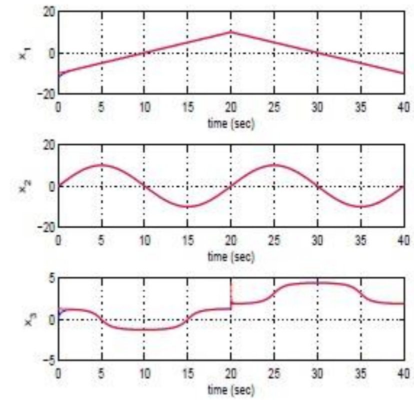
Example 3: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

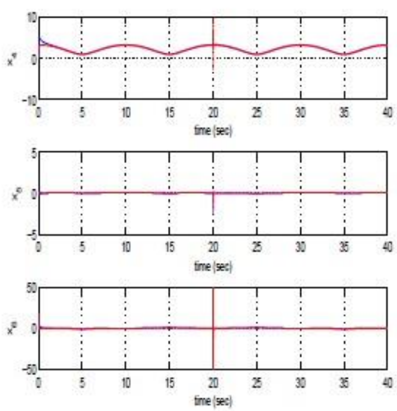
Path 4



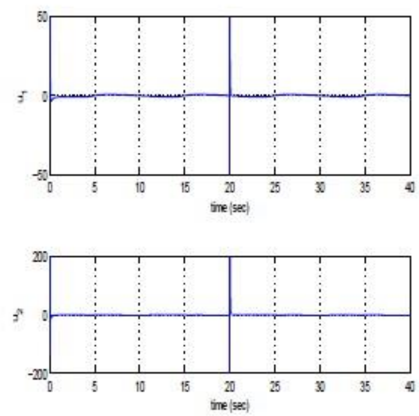
Tracking of the reference trajectory (red line) in the x − y plane by the unmanned surface vessel (blue line),



Convergence of the state variables x1 = x, x2 = y and x3 = ψ to the reference values



Convergence of the state variables of the vessel x4 = u, x5 = v and x6 = r to the reference values

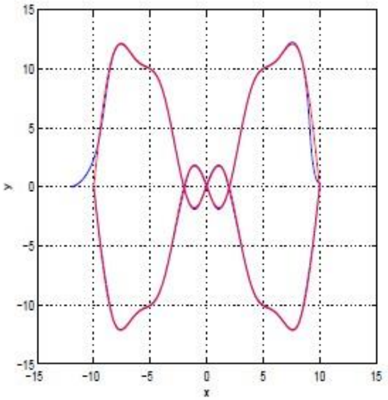


Control inputs u1 and u2 exerted on vessel

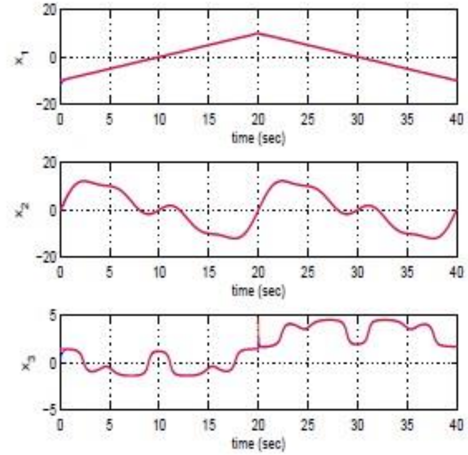
Example 3: Nonlinear control and state estimation using approximate linearization

8. Simulation tests

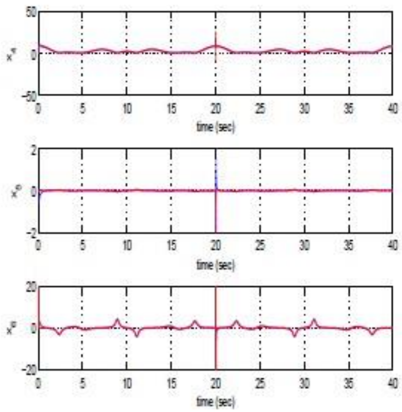
Path 5



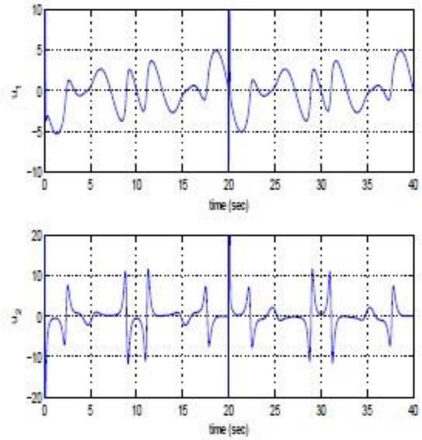
Tracking of the reference trajectory (red line) in the x − y plane by the unmanned surface vessel (blue line),



Convergence of the state variables x1 = x, x2 = y and x3 = ψ to the reference values



Convergence of the state variables of the vessel x4 = u, x5 = v and x6 = r to the reference values



Control inputs u1 and u2 exerted on vessel

Example 3: Nonlinear control and state estimation using approximate linearization

8. Conclusions

• The problem of **trajectory tracking control** of underactuated **USVs** has been solved with a nonlinear H-infinity (optimal) control method.

• A **new nonlinear feedback control method for underactuated vessels** has been developed based on approximate linearization and the use of ***H*-infinity control and stability theory**.

• The first stage of the proposed control method is the **linearization of the unmanned surface vessel** using first order Taylor series expansion and the computation of the associated Jacobian matrices.

• The errors due to the **approximative linearization** have been considered as disturbances that affect, together with external perturbations, the distributed power generators' model.

• At a second stage the implementation of H-infinity feedback control has been proposed. Using the **linearized model of the vehicle** an **H-infinity feedback con**trol law is computed at each iteration of the control algorithm, after previously solving an **algebraic Riccati equation**.

• The known **robustness features of H-infinity control** enable to compensate for the errors of the approximative linearization, as well as to eliminate the effects of external perturbations.

• The **efficiency of the proposed control scheme** is shown analytically and is confirmed through simulation experiments.

Example 4: Nonlinear control and state estimation using approximate linearization

1 . Control of a submarine's diving

● A nonlinear H-infinity (optimal) control method is developed for the problem of simultaneous **control of the depth and heading angle** of an **autonomous submarine**.

● This is a **multi-variable nonlinear control problem** and its solution allows for **precise underwater navigation** of the submarine.

● The **submarine's dynamic model** undergoes approximate linearization around a **temporary operating point** that is recomputed at each step of the control algorithm.

● The linearization procedure is based on **Taylor series expansion** and on the computation of the submarine's model **Jacobian matrices**.

● For the approximately linearized model, the **optimal control problem is solved** through the design of an **H-infinity feedback controller**.

● The computation of the **controller's gains** requires the solution of an **algebraic Riccati equation**, which is repetitively performed at each step of the control method.

● The **stability properties of the control scheme** is proven through Lyapunov analysis. It is shown that that the control scheme is **globally asymptotically stable**.

Example 4: Nonlinear control and state estimation using approximate linearization

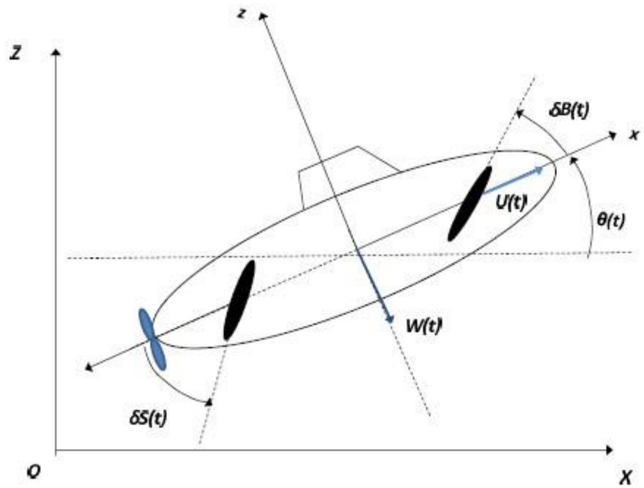
2 . Problem statement

The **multivariable model** of the **submarine's dynamics** has as outputs

the **depth** of the submarine h
The **pitch angle** of the submarine θ

and as inputs

the **deflection angle of the hydroplanes** at the **front part** of vessel δB
the **deflection angle of the hydroplanes** at the **rear** part of the vessel δS



The objective is to achieve **control of the submarine's diving** through **the** for the solution of the associated **nonlinear optimal control** problem

Example 4: Nonlinear control and state estimation using approximate linearization

3 . Dynamic model of the submarine

The **dynamic model of the submarine** is written as:

$$\dot{w}(t) = \frac{Z'_w U}{L m'_3} w(t) + \frac{1}{m'_3} \dot{Z}'_{\dot{\theta}} + m') U \dot{\theta}(t) + \frac{Z'_{\dot{Q}} L}{m'_3} \dot{Q}(t) + $$
$$+ \frac{Z'_{\delta B} U^2}{m'_3 L} \delta B(t) + \frac{Z'_{\delta S} U^2}{m'_3 L} \delta S(t) + \frac{Z_d(t)}{0.5 \rho L^3 m'_3} + Z_\eta(w, q) \quad (1)$$

$$\dot{Q}(t) = \frac{M'_{\dot{w}}}{L I'_2} \dot{w}(t) + \frac{M'_w U}{L^2 I'_2} w(t) + \frac{M'_{\dot{\theta}} U}{L I'_2} \dot{\theta}(t) + $$
$$+ \frac{M'_{\delta B} U^2}{L^2 I'_2} \delta B(t) + \frac{M'_{\delta S} U^2}{L^2 I'_2} \delta S(t) + \frac{2 m g (z_G - z_B)}{\rho L^5 I'_2} \theta(t) + \frac{M_d(t)}{0.5 \rho L^5 I'_2} + M_\eta(w, q) \quad (2)$$

w is the velocity along the z-axis, of the body-fixed frame

h is the depth of the vessel measured in the inertial coordinates system,

θ is the pitch angle

$Q = \dot{\theta}$ is the rate of change of the pitch angle

δB is the hydroplane deflection in the bow plane,

δS is the hydroplane deflection in the stern

Z_d, M_d are bounded disturbance inputs due to sea currents

$Z_\eta(w, q)$, $M_\eta(w, q)$ are disturbance inputs representing the vessel's cross-flow drag

$U = U_0$ denotes the x-axis (forward) velocity of the vessel.

Example 4: Nonlinear control and state estimation using approximate linearization

3 . Dynamic model of the submarine

Indicative values of the parameters of the submarine's dynamic model are:

Table I

Parameters of the Submarine's dynamic model

Parameter Value	Parameter Value	Parameter Value
$Z'_w = -0.0110$	$Z'_{\dot{w}} = -0.0075$	$Z'_\theta = -0.0045$
$Z'_\theta = -0.0002$	$Z'_{\delta B} = -0.0025$	$Z'_{\delta S} = -0.0050$
$M'_w = 0.0030$	$M'_{\dot{w}} = -0.0002$	$M'_\theta = -0.0025$
$M'_{\dot{\theta}} = -0.0004$	$M'_{\delta B} = 0.0005$	$M'_{\delta S} = -0.0025$
$I'_y = 5.6867 \cdot 10e^{-4}$	$L = 286\text{ft}$	$m = 1.52 \cdot 10^5 \text{slug}$
$Z_g - Z_B = -1.5\text{ft}$	$U = 8.43\text{ft/s}$	$\rho = 2.0\text{slug}/\text{ft}^3$
$I'_2 = I'_y - M'_B$	$m = 2m/(\rho L^3))$	$m'_3 = m' - Z'_w$

These can be obtained directly from the design characteristics of the vessel or indirectly through an **identification procedure** in the sense of nonlinear least squares or nonlinear Kalman Filtering

Even in the case that the values of these parameters are known within uncertainty ranges the proposed control method is sufficiently robust to compensate for such a type of model imprecision .

The proposed nonlinear optimal control assures stability of the control loop under parametric changes and and unknown **external perturbations**..

Example 4: Nonlinear control and state estimation using approximate linearization

3. Dynamic model of the submarine

The **dynamic model of the submarine** can be written in **matrix form**:

$$\begin{pmatrix} \dot{w} \\ \dot{Q} \end{pmatrix} = \begin{pmatrix} f_W(w,\theta,Q,t) \\ f_\theta(w,\theta,Q,t) \end{pmatrix} + B_o u \qquad (3)$$

where the **control input vector** is: $u = [\delta B(t) \;\; \delta S(t)]^T$

and is generated by **electric actuators** that rotate the hydroplanes. Therefore the control input describes actually **voltage or current signals** that define the turn angle of the rotor of these electric actuators.

In this description:

$$\begin{pmatrix} f_w(w,\theta,Q,t) \\ f_\theta(f_W(w,\theta,Q,t) \end{pmatrix} = M^{-1} \begin{pmatrix} \frac{Z_w' U}{L m_3'} w(t) + \frac{1}{m_3'} \dot{Z}_\theta' + m')U\dot{\theta}(t) + \frac{Z_{\dot{Q}}' L}{m_s'} \dot{Q}(t) + \frac{Z_d(t)}{0.5\rho L^3 m_3'} + Z_\eta(w,q) \\ \frac{M_{\dot{w}}'}{L I_o'} \dot{w}(t) + \frac{M_v' U}{L^2 I_o'} w(t) + \frac{M_{\dot{\theta}}' U}{L I_o'} \dot{\theta}(t) + \frac{2mg(z_G - z_B)}{\rho L^5 I_o'} \theta(t) + \frac{M_d(t)}{0.5\rho L^5 I_o'} + M_\eta(w,q) \end{pmatrix}$$

while for matrices M and B_o it holds

$$M = \begin{pmatrix} 1 & -Z_{\dot{Q}} L / m_3' \\ -M_{\dot{w}}(L I_2'^{-1}) & 1 \end{pmatrix} \qquad B_o = \begin{pmatrix} \frac{Z_{\delta B}' U^2}{m_3' L} & \frac{Z_{\delta S}' U^2}{m_3' L} \\ \frac{M' \delta B U^2}{L^2 I_2'} & \frac{M' \delta S U^2}{L^2 I_2'} \end{pmatrix}$$

Example 4: Nonlinear control and state estimation using approximate linearization

3. Dynamic model of the submarine

It holds that the **depth of the vessel** measured **in the inertial reference frame** and the velocity **w** of the submarine along the z-axis of the **body-fixed frame** are related as follows:

$$\dot{h} = w cos(\theta) - U_o sin(\theta) \Rightarrow$$
$$\ddot{h} = \dot{w} cos(\theta) - w sin(\theta)\dot{\theta} - U_o cos(\theta)\dot{\theta} \Rightarrow$$
$$\ddot{h} = \dot{w} cos(\theta) - wQ sin(\theta) - U_o Q cos(\theta) \quad (4)$$

From the above relation one can compute about the **diving speed** of the vessel:

$$w = (cos(\theta)^{-1})(\dot{h} + U_o sin(\theta)) \quad (5)$$

Moreover, from Eq (3) one has:

$$\dot{w} = f_w(w,\theta,Q,t) + B_{o_{11}}u_1 + B_{o_{12}}u_2$$
$$\dot{Q} = f_\theta(w,\theta,Q,t) + B_{o_{21}}u_1 + B_{o_{22}}u_2 \quad (6)$$

Substituting Eq. (5) and the first row of Eq. (6) into Eq. (4) one gets

$$\ddot{h} = [f_w(w,\theta,Q,t) + B_{o_{11}}u_1 + B_{o_{12}}u_2]cos(\theta) - \frac{(\dot{h}+U_0 sin(\theta))}{cos(\theta)}Q sin(\theta) - U_0 Q cos(\theta) \quad (7)$$

Example 4: Nonlinear control and state estimation using approximate linearization

3. Dynamic model of the submarine

Next, by denoting:

$$f_w(w,\theta,Q,t) = g_h(h,\dot{h},\theta,\dot{\theta},t)$$
$$f_\theta(w,\theta,Q,t) = g_\theta(h,\dot{h},\theta,\dot{\theta},t)$$

And by substituting this relation in Eq. (7), together with $Q=\dot{\theta}$ one obtains:

$$\ddot{h} = g_h(h,\dot{h},\theta,\dot{\theta},t)cos(\theta) - \frac{(\dot{h}+U_0 sin(\theta))}{cos(\theta)}\dot{\theta}sin(\theta) - U_0\dot{\theta}cos(\theta)+$$
$$+B_{0_{11}}cos(\theta)u_1 + B_{0_{12}}cos(\theta)u_2$$

$$\ddot{\theta} = g_\theta(h,\dot{h},\theta,\dot{\theta},t) + B_{0_{21}}u_1 + B_{0_{22}}u_2$$

Then, by defining the **state vector** $x=[h,\dot{h},\theta,\dot{\theta}]^T$

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_3\end{pmatrix} = \begin{pmatrix} g_b(x,t)cos(x_3) - \frac{x_4+U_0 sin(x_3)}{cos(x_3)}x_4 sin(x_3) - U_0 x_4 cos(x_3)\\ g_\theta(x,t)\end{pmatrix} + \begin{pmatrix}B_{0_{11}} & B_{0_{12}}\\ B_{0_{21}} & B_{0_{22}}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} \quad (8)$$

From Eq. (8) one finally arrives at the **MIMO state-space description** of the submarine

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_3\end{pmatrix} = \begin{pmatrix}f_1(x,t)\\ f_2(x,t)\end{pmatrix} + \begin{pmatrix}g_{11}(x,t) & g_{12}(x,t)\\ g_{21}(x,t) & g_{22}(x,t)\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} \quad (9)$$

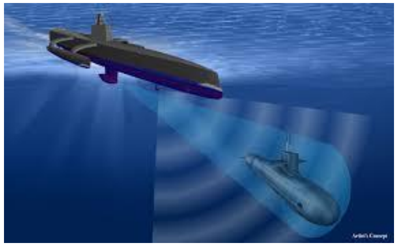
Example 4: Nonlinear control and state estimation using approximate linearization

4. Approximate linearization of the submarine's dynamics

Using the previous description of the submarine's dynamics given in Eq. (8) **one has that**

$$\begin{pmatrix} g_h(x,t) \\ g_\theta(x,t) \end{pmatrix} = \begin{pmatrix} 1 & -Z_Q' L/m_3' \\ -M_{\ddot{w}}(LI_2'^{-1}) & 1 \end{pmatrix}^{-1} \cdot \quad (10)$$

$$\begin{pmatrix} \frac{Z_w' U}{L m_3'} w(t) + \frac{1}{m_3'}(\dot{Z}_{\dot{\theta}}' + m')U\dot{\theta}(t) + \frac{Z_d(t)}{0.5\rho L^3 m_3'} + Z_\eta(w,Q) \\ \frac{M_v' U}{L^2 I_2'} w(t) + \frac{M_{\dot{\theta}}' U}{L I_2'}\dot{\theta}(t) + \frac{2mg(z_G - z_B)}{\rho L^5 I_2'}\theta(t) + \frac{M_d(t)}{0.5\rho L^5 I_2'} + M_\eta(w,Q) \end{pmatrix}$$



The effects of the wave and currents forces and of hydrodynamic forces are considered as disturbances and are not given explicitly in the model of the submarine's dynamics.

By grouping coefficients the previous equation given in Eq. (10) can be written as

$$\begin{pmatrix} g_h(x,t) \\ g_\theta(x,t) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} \frac{1}{cos(x_3)}[x_2 + U_0 sin(x_3)] \\ x_4 \end{pmatrix} \quad (11)$$

and by performing additional operations between coefficients one has

$$\begin{pmatrix} g_h(x,t) \\ g_\theta(x,t) \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{cos(x_3)}[x_2 + U_0 sin(x_3)] \\ x_4 \end{pmatrix}$$

Example 4: Nonlinear control and state estimation using approximate linearization

4. Approximate linearization of the submarine's dynamics

According to the above, the AUV's model is written in the generic form:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} + \begin{pmatrix} G_{11}(x) \\ G_{21}(x) \end{pmatrix} u_1 + \begin{pmatrix} G_{12}(x) \\ G_{22}(x) \end{pmatrix} u_2 \quad (12)$$

or equivalently

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ F_1(x) \\ x_4 \\ F_2(x) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_{11}(x) & G_{12}(x) \\ 0 & 0 \\ G_{21}(x) & G_{22}(x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where one has that

$$F_1(x) = p_{11}\frac{1}{cos(x_3)}(x_2 + U_0 sin(x_3)) + p_{12}x_4 - \frac{x_2 + U_0 sin(x_3)}{cos(x_3)} x_4 sin(x_3) - U_0 x_4 sin(x_3)$$

$$F_2(x) = p_{21}\frac{1}{cos(x_3)}(x_2 + U_0 sin(x_3)) + p_{22}x_4 \quad (13)$$

while it also holds that

$$G_{11}(x) = B_{011}cos(x_3) \quad G_{12}(x) = B_{012}cos(x_3)$$
$$G_{21}(x) = B_{021} \quad G_{22}(x) = B_{022} \quad (14)$$

Example 4: Nonlinear control and state estimation using approximate linearization

4. Approximate linearization of the submarine's dynamics

Next, the Jacobian matrices of the submarine's dynamic model are computed. For the Jacobian matrix $\nabla_x F$ one has:

$$\nabla_x F = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} & \frac{\partial F_3}{\partial x_4} \\ \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} & \frac{\partial F_4}{\partial x_3} & \frac{\partial F_4}{\partial x_4} \end{pmatrix} \quad (15)$$

For the Jacobian matrix $\nabla_x G_1$ one has:

$$\nabla_x G_1 = \begin{pmatrix} \frac{\partial G_{11}}{\partial x_1} & \frac{\partial G_{11}}{\partial x_2} & \frac{\partial G_{11}}{\partial x_3} & \frac{\partial G_{11}}{\partial x_4} \\ \frac{\partial G_{21}}{\partial x_1} & \frac{\partial G_{21}}{\partial x_2} & \frac{\partial G_{21}}{\partial x_3} & \frac{\partial G_{21}}{\partial x_4} \\ \frac{\partial G_{31}}{\partial x_1} & \frac{\partial G_{31}}{\partial x_2} & \frac{\partial G_{31}}{\partial x_3} & \frac{\partial G_{31}}{\partial x_4} \\ \frac{\partial G_{41}}{\partial x_1} & \frac{\partial G_{41}}{\partial x_2} & \frac{\partial G_{41}}{\partial x_3} & \frac{\partial G_{41}}{\partial x_4} \end{pmatrix} \quad (16)$$

For the Jacobian matrix $\nabla_x G_2$ one has:

$$\nabla_x G_2 = \begin{pmatrix} \frac{\partial G_{12}}{\partial x_1} & \frac{\partial G_{12}}{\partial x_2} & \frac{\partial G_{12}}{\partial x_3} & \frac{\partial G_{12}}{\partial x_4} \\ \frac{\partial G_{22}}{\partial x_1} & \frac{\partial G_{22}}{\partial x_2} & \frac{\partial G_{22}}{\partial x_3} & \frac{\partial G_{22}}{\partial x_4} \\ \frac{\partial G_{32}}{\partial x_1} & \frac{\partial G_{32}}{\partial x_2} & \frac{\partial G_{32}}{\partial x_3} & \frac{\partial G_{32}}{\partial x_4} \\ \frac{\partial G_{42}}{\partial x_1} & \frac{\partial G_{42}}{\partial x_2} & \frac{\partial G_{42}}{\partial x_3} & \frac{\partial G_{42}}{\partial x_4} \end{pmatrix} \quad (17)$$

Example 4: Nonlinear control and state estimation using approximate linearization

5. Design of an H-infinity controller for the submarine’s model

As explained, the system’s dynamic model undergoes **linearization** round its present operating point (*x**,*u**), where *x** is the present value of the submarine system’s state vector and *u** is the last sampled value of the control inputs vector..

Thus one arrives at the **approximately linearized description** of the system:

$$\dot{x} = Ax + Bu + \tilde{d} \quad (18)$$

where d_1 is the linearization error due to truncation of higher-order terms in the **Taylor series expansion** and

$$A = [\nabla_x F + \nabla_x G_1 u_1 + \nabla_x G_2 u_2] \mid_{(x^*,u^*)} \quad (19)$$

In a similar manner, one has that

$$B = [\nabla_u F + \nabla_u G_1 u_1 + \nabla_u G_2 u_2] \mid_{(x^*,u^*)} = [G_1, G_2] \quad (20)$$

After **linearization** round its current operating point the system’s **model** is written as

$$\dot{x} = Ax + Bu + d_1 \quad (21)$$

Parameter d_1 stands for the **linearization error** in the system’s model

At every time instant the control input u^* is assumed to differ from the control input u appearing in (21) by an amount equal to Δu that is $u^* = u + \Delta u$

$$\dot{x}_d = Ax_d + Bu^* + d_2 \quad (22)$$

Example 4: Nonlinear control and state estimation using approximate linearization

5. Design of an H-infinity controller for the submarine's model

The dynamics of the system of Eq. (21) can be also written in the form

$$\dot{x} = Ax + Bu + Bu^* - Bu^* + d_1 \quad (23)$$

and by denoting $d_3 = -Bu^* + d_1$ as an **aggregate disturbance** term one obtains

$$\dot{x} = Ax + Bu + Bu^* + d_3 \quad (24)$$

By subtracting Eq. (24) from Eq. (21) one has

$$\dot{x} - \dot{x}_d = A(x - x_d) + Bu + d_3 - d_2 \quad (25)$$

By denoting the tracking error as $e = x - x_d$ and the aggregate disturbance term as $\tilde{d} = \tilde{d}_3 - \tilde{d}_2$ the **tracking error dynamics** becomes

$$\dot{e} = Ae + Bu + \tilde{d} \quad (26)$$

Example 4: Nonlinear control and state estimation using approximate linearization

5. Design of an H-infinity controller for the submarine's model

The initial model of the submarine is assumed to be in the form

$$\dot{x} = f(x,u) \quad x \in R^n, \ u \in R^m$$

where the **linearization point** is defined by the present value of the system's state vector and the last sampled value of the control inputs vector

$$(x^*, u^*) = (x(t), u(t - T_s)).$$

The **linearized equivalent of the system** is described by

$$\dot{x} = Ax + Bu + Ld \quad x \in R^n, \ u \in R^m, \ d \in R^q$$

where matrices A and B are obtained from the **computation of the Jacobians**

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \Big|_{(x^*,u^*)} \qquad B = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{pmatrix} \Big|_{(x^*,u^*)}$$

and vector d denotes disturbance terms due to linearization errors.

$$\dot{x} = Ax + Bu + Ld$$
$$y = Cx$$

Example 3: Nonlinear control and state estimation using approximate linearization

6. Lyapunov stability analysis

The **tracking error dynamics** for the submarine's model is written in the form

$$\dot{e} = Ae + Bu + L\tilde{d} \quad (27)$$

where in the **case of the considered submarine model** $L = I \in R^4$ with I being the identity matrix. The following **Lyapunov function** is considered

$$V = \frac{1}{2}e^T P e \quad (28)$$

where $e = x - x_d$ Is the state vector's tracking error

$$\dot{V} = \frac{1}{2}\dot{e}^T P e + \frac{1}{2}e^T P \dot{e} \Rightarrow$$

$$\dot{V} = \frac{1}{2}[Ae + Bu + L\tilde{d}]^T P + \frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow$$

$$\dot{V} = \frac{1}{2}[e^T A^T + u^T B^T + \tilde{d}^T L^T]Pe + \\ + \frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow$$

$$\dot{V} = \frac{1}{2}e^T A^T Pe + \frac{1}{2}u^T B^T Pe + \frac{1}{2}\tilde{d}^T L^T Pe + \\ \frac{1}{2}e^T PAe + \frac{1}{2}e^T PBu + \frac{1}{2}e^T PL\tilde{d}$$

Example 4: Nonlinear control and state estimation using approximate linearization

6. Lyapunov stability analysis

The previous equation is rewritten as

$$\dot{V}=\frac{1}{2}e^T(A^TP+PA)e+(\frac{1}{2}u^TB^TPe+\frac{1}{2}e^TPBu)+ \\ +(\frac{1}{2}\tilde{d}^TL^TPe+\frac{1}{2}e^TPL\tilde{d})$$

Assumption: For given positive definite matrix Q *and coefficients* r and ρ there exists a positive definite matrix P, which is the solution of the following matrix equation

$$A^TP+PA=-Q+P(\frac{2}{r}BB^T-\frac{1}{\rho^2}LL^T)P \quad (29)$$

Moreover, the following **feedback control law** is applied to the PEM fuel cells model

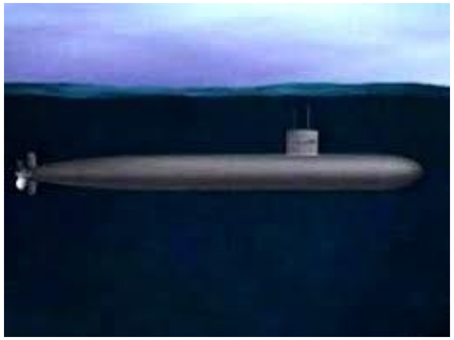
$$u=-\frac{1}{r}B^TPe \quad (30)$$

By substituting Eq. (29) and Eq. (30) one obtains

$$\dot{V}=\frac{1}{2}e^T[-Q+P(\frac{2}{r}BB^T-\frac{1}{\rho^2}LL^T)P]e+ \\ +e^TPB(-\frac{1}{r}B^TPe)+e^TPL\tilde{d}\Rightarrow$$

Example 4: Nonlinear control and state estimation using approximate linearization

6. Lyapunov stability analysis



Continuing with computations one obtains

$$\dot{V} = -\frac{1}{2}e^TQe + \frac{1}{r}e^TPBB^TPe - \frac{1}{2\rho^2}e^TPLL^TPe$$
$$-\frac{1}{r}e^TPBB^TPe + e^TPL\tilde{d}$$

which next gives

$$\dot{V} = -\frac{1}{2}e^TQe - \frac{1}{2\rho^2}e^TPLL^TPe + e^TPL\tilde{d}$$

or equivalently

$$\dot{V} = -\frac{1}{2}e^TQe - \frac{1}{2\rho^2}e^TPLL^TPe+$$
$$+\frac{1}{2}e^TPL\tilde{d} + \frac{1}{2}\tilde{d}^TL^TPe \quad (31)$$

Lemma: The following inequality holds

$$\frac{1}{2}e^TL\tilde{d} + \frac{1}{2}\tilde{d}L^TPe - \frac{1}{2\rho^2}e^TPLL^TPe \leq \frac{1}{2}\rho^2\tilde{d}^T\tilde{d}$$

Example 4: Nonlinear control and state estimation using approximate linearization

6. Lyapunov stability analysis

Proof : *The binomial* $(\rho a - \frac{1}{\rho}b)^2$ is considered. Expanding the left part of the above inequality one gets

$$\rho^2 a^2 + \frac{1}{\rho^2}b^2 - 2ab \ge 0 \Rightarrow \frac{1}{2}\rho^2 a^2 + \frac{1}{2\rho^2}b^2 - ab \ge 0 \Rightarrow$$
$$ab - \frac{1}{2\rho^2}b^2 \le \frac{1}{2}\rho^2 a^2 \Rightarrow \frac{1}{2}ab + \frac{1}{2}ab - \frac{1}{2\rho^2}b^2 \le \frac{1}{2}\rho^2 a^2$$

The following substitutions are carried out: $a = \tilde{d}$ and $b = e^T P L$ and the previous relation becomes

$$\frac{1}{2}\tilde{d}^T L^T P e + \frac{1}{2}e^T P L \tilde{d} - \frac{1}{2\rho^2}e^T P L L^T P e \le \frac{1}{2}\rho^2 \tilde{d}^T \tilde{d} \qquad (32)$$

Eq. (32) is substituted in Eq. (31) and the inequality is enforced, thus giving

$$\dot{V} \le -\frac{1}{2}e^T Q e + \frac{1}{2}\rho^2 \tilde{d}^T \tilde{d} \qquad (33)$$

Eq. (33) shows that the **H-infinity tracking performance criterion** is satisfied.

The integration of $\dot{V}$ from 0 to T gives

$$\int_0^T \dot{V}(t)dt \le -\frac{1}{2}\int_0^T ||e||_Q^2 dt + \frac{1}{2}\rho^2 \int_0^T ||\tilde{d}||^2 dt \Rightarrow$$
$$2V(T) + \int_0^T ||e||_Q^2 dt \le 2V(0) + \rho^2 \int_0^T ||\tilde{d}||^2 dt$$

Example 4: Nonlinear control and state estimation using approximate linearization

6. Lyapunov stability analysis

Moreover, if there exists a positive constant $M_d > 0$ such that

$$\int_0^{\infty} ||\tilde{d}||^2 dt \leq M_d$$

then one gets

$$\int_0^{\infty} ||e||_Q^2 dt \leq 2V(0) + \rho^2 M_d$$

Thus, the integral $\int_0^{\infty} ||e||_Q^2 dt$ is bounded.

Moreover, V(T) is bounded and from the definition of the Lyapunov function V it becomes clear that **e(t) will be also bounded** since

$$e(t) \in \Omega_e = \{e | e^T P e \leq 2V(0) + \rho^2 M_d\}.$$

According to the above and with the use of **Barbalat's Lemma** one obtains:

$$lim_{t \to \infty} e(t) = 0.$$

34

Example 4: Nonlinear control and state estimation using approximate linearization

7. Robust state estimation with the use of the H-infinity Kalman Filter

- The control loop has to be implemented with the use of information provided by a **small number of measurements** of the state variables of the submarine's model

- To reconstruct the missing information about the state vector of the submarine's model it is proposed to **use a filter** and based on it to apply state **estimation-based control** .

- The **recursion of the H-infinity Kalman Filter**, for the **submarine's model**, can be formulated in terms of a measurement update and a time update part

Measurement update

$$D(k) = [I - \theta W(k) P^-(k) + C^T(k) R(k)^{-1} C(k) P^-(k)]^{-1}$$
$$K(k) = P^-(k) D(k) C^T(k) R(k)^{-1}$$
$$\hat{x}(k) = \hat{x}^-(k) + K(k)[y(k) - C\hat{x}^-(k)]$$

Time update

$$\hat{x}^-(k+1) = A(k) x(k) + B(k) u(k)$$
$$P^-(k+1) = A(k) P^-(k) D(k) A^T(k) + Q(k)$$

where it is assumed that parameter θ is sufficiently small to assure that the **covariance matrix**

$$P^-(k)^{-1} - \theta W(k) + C^T(k) R(k)^{-1} C(k)$$

Is **positive definite**

Example 4: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

- The performance of the proposed nonlinear **H-nfinity control scheme** for the **submarine's model** is tested through simulation:

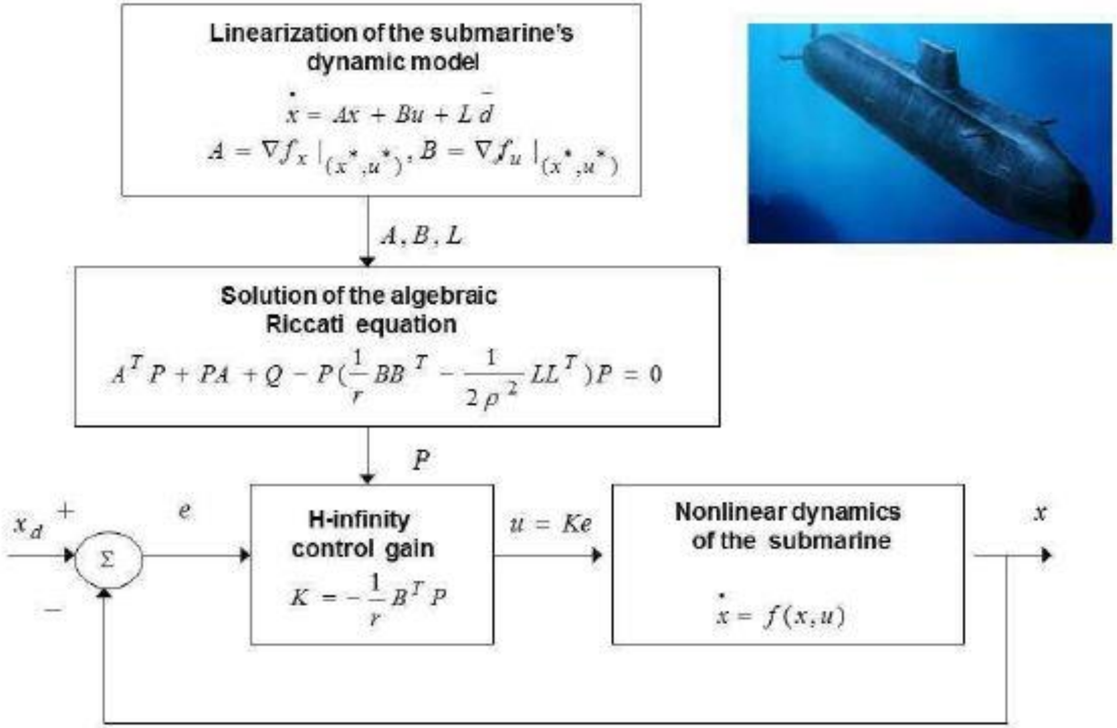


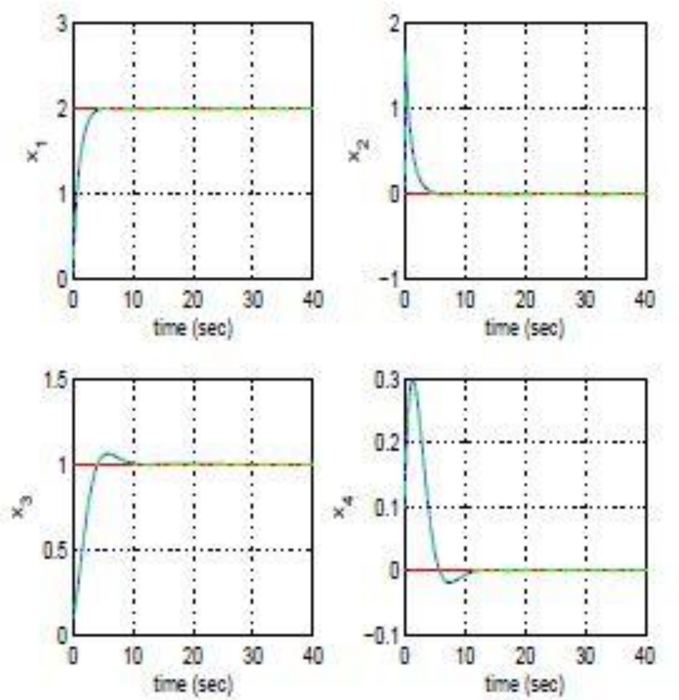
Fig. 1 Diagram of the nonlinear optimal control for the diving submarine

With the use of the proposed H-infinity control method, fast and accurate tracking of the reference setpoints of the **submarine's model state variables** was achieved

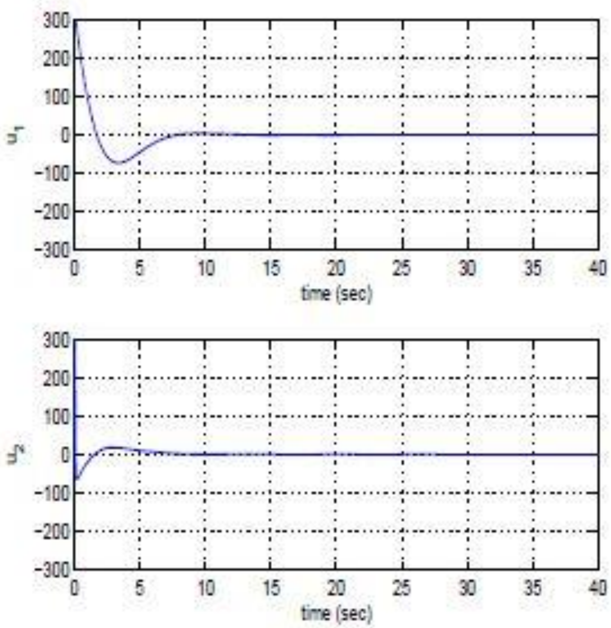
Example 4: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

- Out of the 4 state variables of the **autonomous submarine** only 2 where considered to be measurable. These were the submarine's depth h and the its heading angle θ



Tracking of setpoint 1: (a) Convergence of the state variables (blue lines) to setpoints (red lines) and their state estimates (green lines)

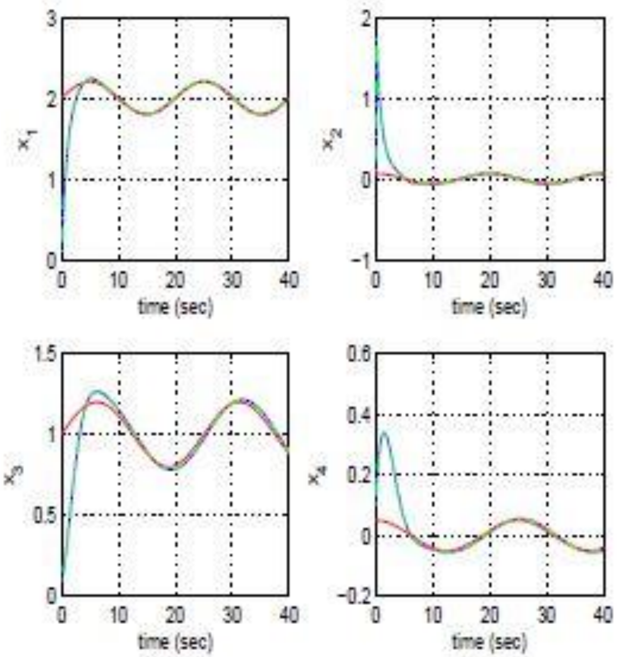


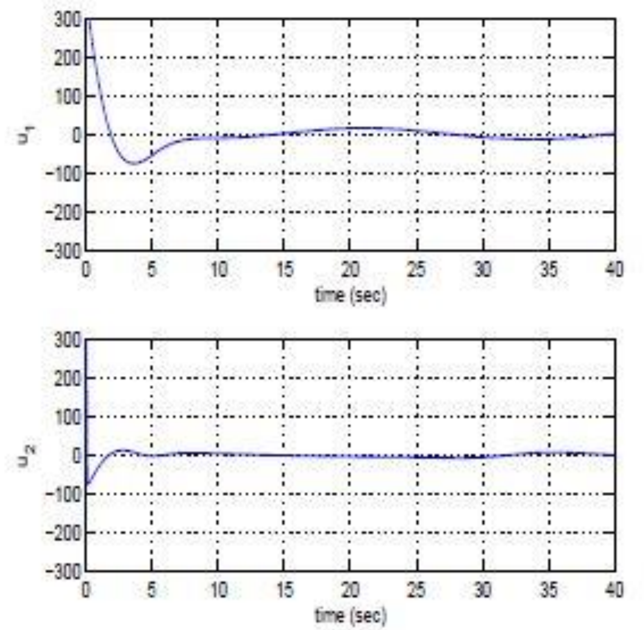
Tracking of setpoint 1: (b) variation of the submarine's control inputs

Example 4: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

● The use of the H-infinity Kalman Filter as a robust state estimator allows for implementing feedback control based on a small number of sensors and measuring equipment of the submarine





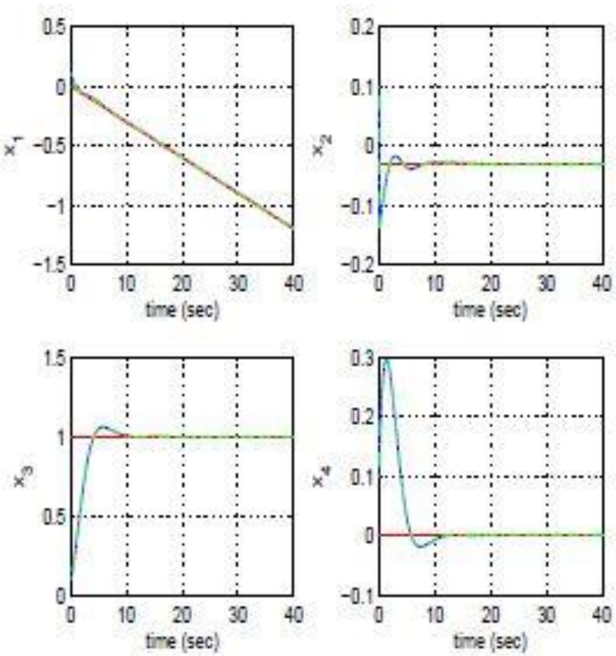
Tracking of setpoint 2: (a) Convergence of the state variables (blue lines) to setpoints (red lines) and their state estimates (green lines)

Tracking of setpoint 2: (b) variation of the submarine's control inputs

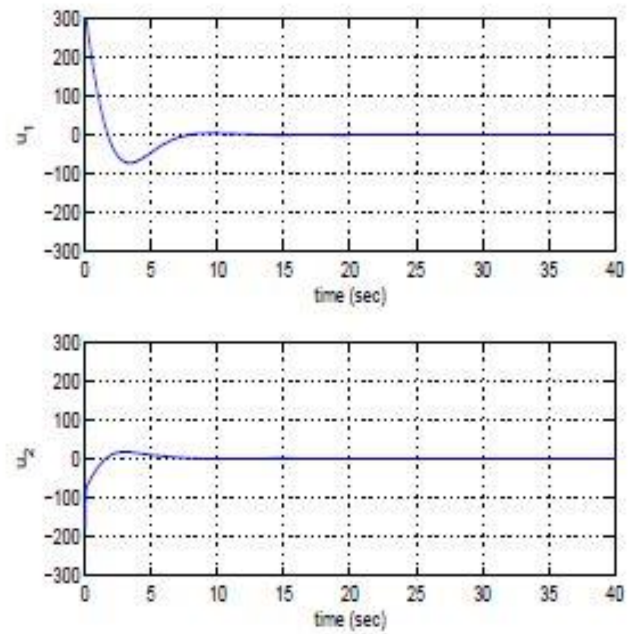
Example 4: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

- For the computation of the feedback control gain the algebraic Riccati equation appearing in Eq. (29) had to be repetitively solved at each step of thecontrol method.



Tracking of setpoint 3: (a) Convergence of the state variables (blue lines) to setpoints (red lines) and their state estimates (green lines)

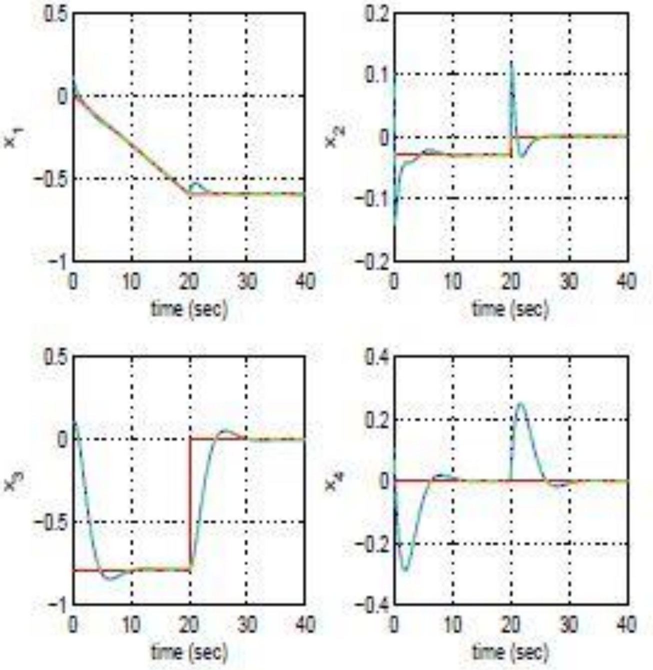


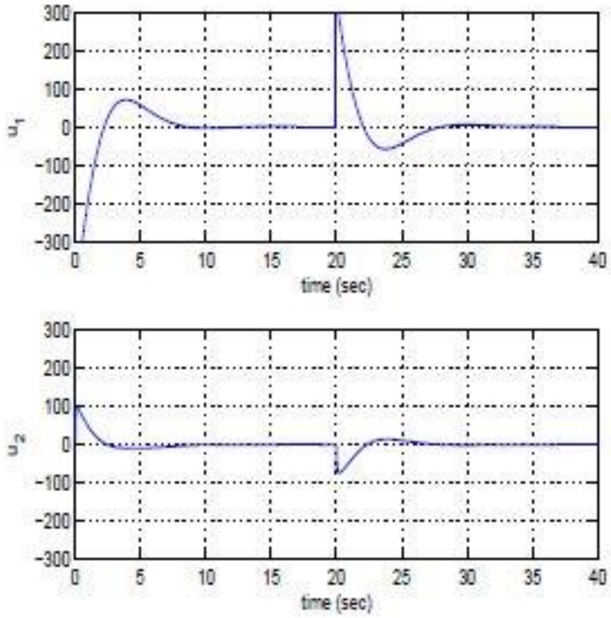
Tracking of setpoint 3: (b) variation of the submarine's control inputs

Example 4: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

- Unlike global linearization-based control methods the proposed nonlinear optimal control is applied directly on the nonlinear dynamical model of the submarine and does not require the computation of diffeomorphisms (change of state variables)





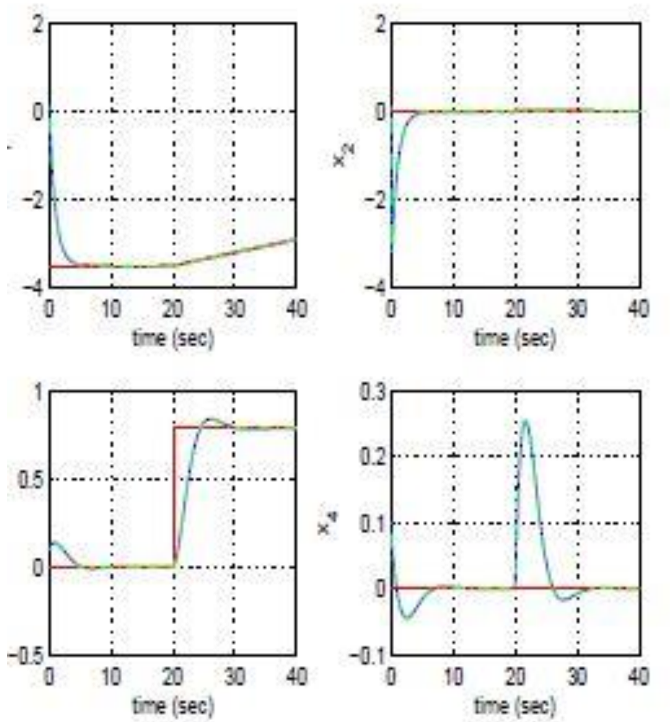
Tracking of setpoint 4: (a) Convergence of the state variables (blue lines) to setpoints (red lines) and their state estimates (green lines)

Tracking of setpoint 4: (b) variation of the submarine's control inputs

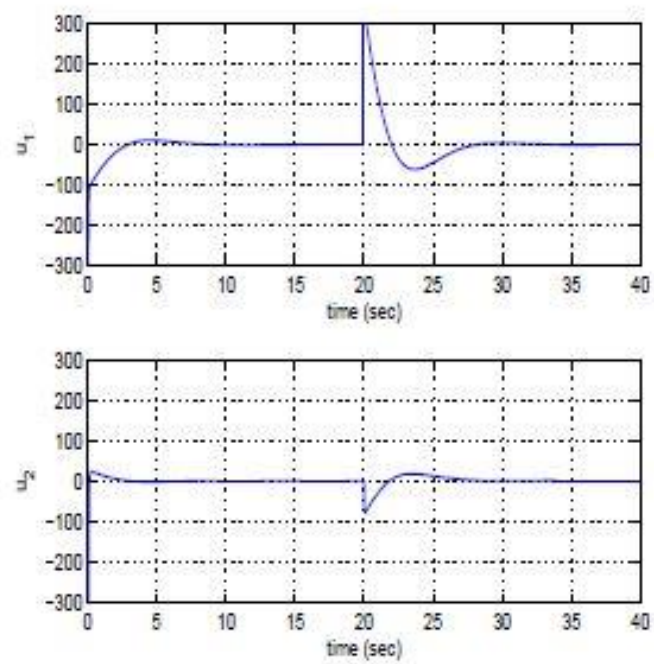
Example 4: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

● The computation of the feedback control signal follows an optimal control concept and retains the advantages of linear optimal control in terms of accuracy of tracking and moderate control inputs variation



Tracking of setpoint 5: (a) Convergence of the state variables (blue lines) to setpoints (red lines) and their state estimates (green lines)

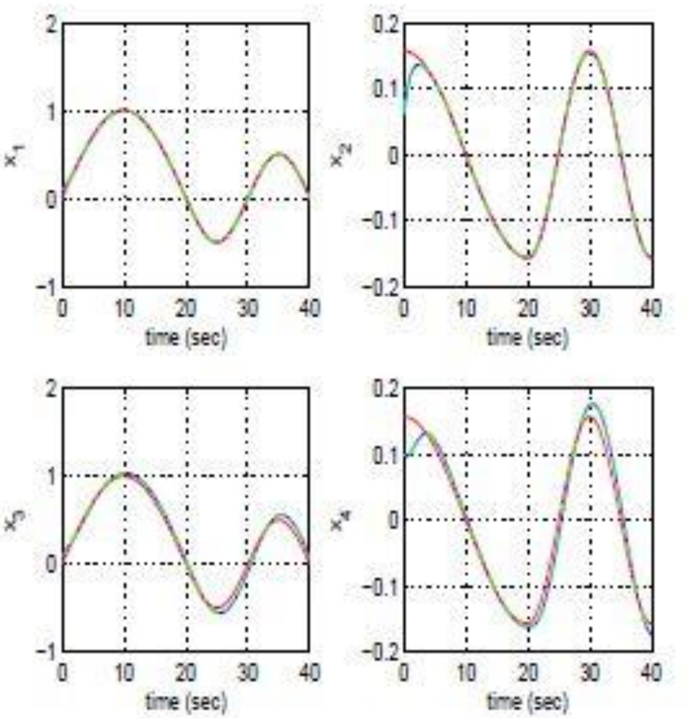


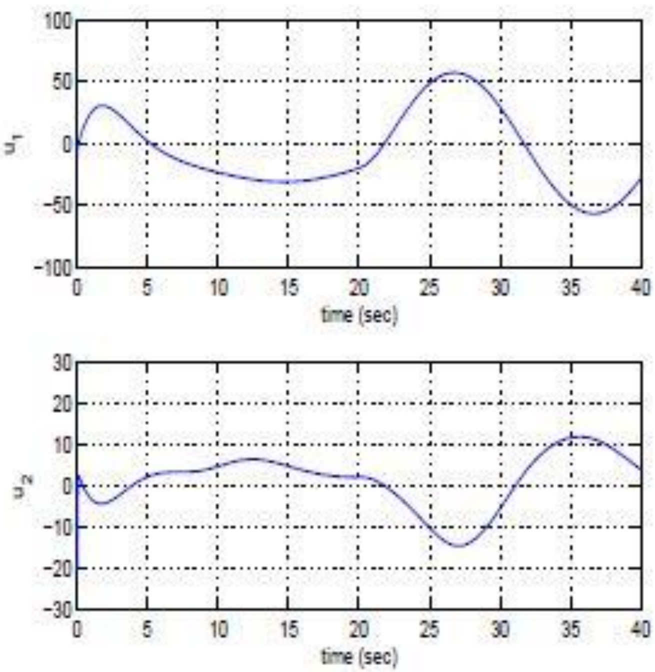
Tracking of setpoint 5: (b) variation of the submarine's control inputs

Example 4: Nonlinear control and state estimation using approximate linearization

7. Simulation tests

● Despite modeling errors induced by the approximate linearization of the Taylor series expansion the proposed control method exhibits significant robustness





Tracking of setpoint 6: (a) Convergence of the state variables (blue lines) to setpoints (red lines) and their state estimates (green lines)

Tracking of setpoint 6: (b) variation of the submarine's control inputs

Example 4: Nonlinear control and state estimation using approximate linearization

8 . Conclusions

● A **nonlinear optimal (H-infinity) control** method has been developed for the control of the **submarine's diving model**.

● The dynamic model of the submarine has undergone first approximate linearization around a temporary operating point which was redefined at each iteration of the control algorithm

●The linearization point (equilibrium) consists at every time instant of the present value of the state vector of the submarine and of the last value of the control inputs vector exerted on it.

● The linearization was based on Taylor series expansion and on the computation of the associated Jacobian matrices. The approximation error was considered to be a disturbance that had to be compensated by the robustness of the control method.

● For the approximately linearized model of the submarine an H-infinity (optimal) feedback controller has been designed.

● The stability features of the submarine's control loop were proven through Lyapunov analysis.. , it was proven that the control loop satisfies also conditions for global asymptotic stability.

Example 5: Nonlinear control and state estimation using Lyapunov methods

1. Control of a submarine's diving

- **Adaptive fuzzy control based** on **differential flatness theory** for **multivariable control** (dive-plane control) of AUVs.

- The **dynamic model of the submarine**, with state variables the vessel's depth and its pitch angle, is a **differentially flat** one. This means that all its state variables and its control inputs can be written as differential functions of the flat output and its derivatives.

- By exploiting differential flatness properties the system's dynamic model is written in the **multivariable linear canonical (Brunovsky) form**, for which the design of a state feedback controller becomes possible.

- After this **transformation**, the **new control in**puts of the system contain **unknown nonlinear parts**, which are **identified with the use of neurofuzzy approximators**.

- The **learning procedure** for these estimators is determined by the requirement the **first derivative of the closed-loop's Lyapunov function to be a negative one**.

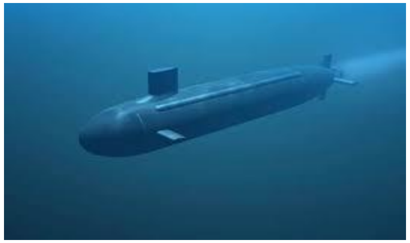
- Moreover, **the Lyapunov stability analysis** shows that **H-infinity tracking performance** is succeeded for the feedback control loop and this assures improved **robustness** to the aforementioned model uncertainty as well as to external perturbations.

- The efficiency of the proposed adaptive fuzzy control scheme isconfirmed through simulation experiments.

Example 5: Nonlinear control and state estimation using Lyapunov methods

2. Problem statement

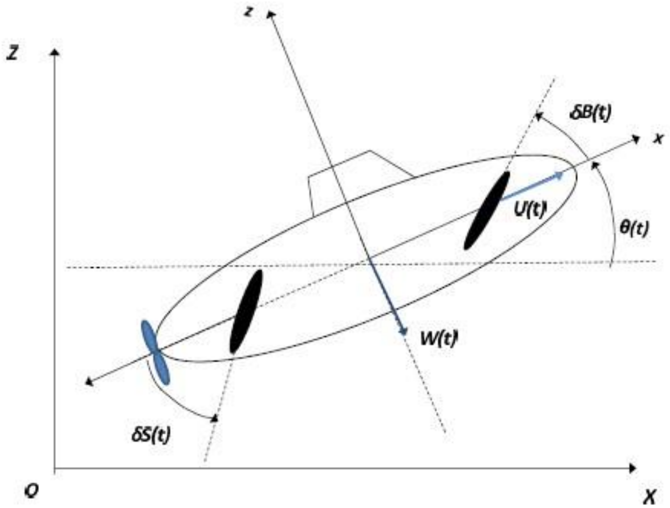
The **multivariable model** of the **submarine's dynamics** has as outputs

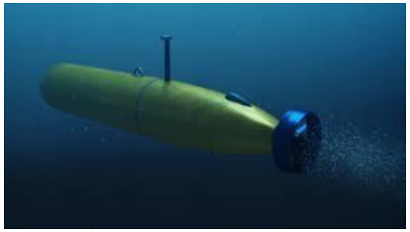


the **depth** of the submarine h
The **pitch angle** of the submarine θ

and as inputs

the **deflection angle of the hydroplanes** at the **front part** of vessel δB
the **deflection angle of the hydroplanes** located at the **rear** part of the vessel δS





The objective is to succeed **multivariable nonlinear feedback control** for the submarine's model, **without prior knowledge of the vessel's kinematic or dynamic model**

Example 5: Nonlinear control and state estimation using Lyapunov methods

3. Dynamic model of the submarine

The **dynamic model of the submarine** is written as:

$$\dot{w}(t) = \frac{Z'_w U}{L m'_3} w(t) + \frac{1}{m'_3}\dot{Z}'_{\dot\theta} + m')U\dot\theta(t) + \frac{Z'_{\dot q} L}{m'_3}\dot{Q}(t) + \frac{Z'_{\delta B}U^2}{m'_3 L}\delta B(t) + \frac{Z'_{\delta S}U^2}{m'_3 L}\delta S(t) + \frac{Z_d(t)}{0.5\rho L^3 m'_3} + Z_\eta(w,q) \quad (1)$$

$$\dot{Q}(t) = \frac{M'_{\dot w}}{L I'_2}\dot{w}(t) + \frac{M'_w U}{L^2 I'_2} w(t) + \frac{M'_{\dot\theta} U}{L I'_2}\dot\theta(t) + \frac{M'_{\delta B}U^2}{L^2 I'_2}\delta B(t) + \frac{M'_{\delta S}U^2}{L^2 I'_2}\delta S(t) + \frac{2mg(z_G - z_B)}{\rho L^5 I'_2}\theta(t) + \frac{M_d(t)}{0.5\rho L^5 I'_2} + M_\eta(w,q) \quad (2)$$

w is the velocity along the z-axis, of the body-fixed frame

h is the depth of the vessel measured in the inertial coordinates system,

θ is the pitch angle

$Q = \dot\theta$ is the rate of change of the pitch angle.

δB is the hydroplane deflection in the bow plane,

δS is the hydroplane deflection in the stern

Z_d, M_d are bounded disturbance inputs due to sea currents

$Z_\eta(w,q)$, $M_\eta(w,q)$ are disturbance inputs representing the vessel's cross-flow drag

$U = U_0$ denotes the x-axis (forward) velocity of the vessel.

Example 5: Nonlinear control and state estimation using Lyapunov methods

3. Dynamic model of the submarine

Indicative values of the parameters of the submarine's dynamic model are:

Table I [1]

Parameters of the Submarine's dynamic model

Parameter Value	Parameter Value	Parameter Value
$Z'_w = -0.0110$	$Z'_{\dot{w}} = -0.0075$	$Z'_\theta = -0.0045$
$Z'_\theta = -0.0002$	$Z'_{\delta B} = -0.0025$	$Z'_{\delta S} = -0.0050$
$M'_w = 0.0030$	$M'_{\dot{w}} = -0.0002$	$M'_\theta = -0.0025$
$M'_{\dot{\theta}} = -0.0004$	$M'_{\delta B} = 0.0005$	$M'_{\delta S} = -0.0025$
$I'_y = 5.6867 \cdot 10e^{-4}$	$L = 286\text{ft}$	$m = 1.52 \cdot 10^5 \text{slug}$
$Z_g - Z_B = -1.5\text{ft}$	$U = 8.43\text{ft/s}$	$\rho = 2.0\text{slug}/\text{ft}^3$
$I'_2 = I'_y - M'_B$	$m = 2m/(\rho L^3))$	$m'_3 = m' - Z'_w$

[1]

These can be obtained directly from the design characteristics of the vessel or indirectly through an **identification procedure** in the sense of nonlinear least squares or nonlinear Kalman Filtering

However, since **adaptive control is a model-free control method**, there is **no need about prior knowledge** of these parameters' values..

Adaptive control assures stability of the control loop **under** unknown dynamic model parameters and unknown **external perturbations and disturbances** ..

Example 5: Nonlinear control and state estimation using Lyapunov methods

3. Dynamic model of the submarine

The **dynamic model of the submarine** can be written in **matrix form**:

$$\begin{pmatrix} \dot{w} \\ \dot{Q} \end{pmatrix} = \begin{pmatrix} f_W(w,\theta,Q,t) \\ f_\theta(w,\theta,Q,t) \end{pmatrix} + B_o u \qquad (3)$$

where the **control input vector** is: $u = [\delta B(t) \;\; \delta S(t)]^T$

and is generated by **electric actuators** that rotate the hydroplanes. Therefore the control input describes actually **voltage or current signals** that define the turn angle of the rotor of these electric actuators.

In this description:

$$\begin{pmatrix} f_w(w,\theta,Q,t) \\ f_\theta(fW(w,\theta,Q,t) \end{pmatrix} = M^{-1} \begin{pmatrix} \frac{Z_w' U}{L m_3'} w(t) + \frac{1}{m_3'} \dot{Z}_\theta' + m') U \dot{\theta}(t) + \frac{Z_{\dot{Q}}' L}{m_z'} \dot{Q}(t) + \frac{Z_d(t)}{0.5 \rho L^3 m_3'} + Z_\eta(w,q) \\ \frac{M_{\dot{w}}'}{L I_0'} \dot{w}(t) + \frac{M_v' U}{L^2 I_0'} w(t) + \frac{M_{\dot{\theta}}' U}{L I_0'} \dot{\theta}(t) + \frac{2 m g (z_G - z_B)}{\rho L^5 I_0'} \theta(t) + \frac{M_d(t)}{0.5 \rho L^5 I_0'} + M_\eta(w,q) \end{pmatrix}$$

while for matrices M and B_o it holds

$$M = \begin{pmatrix} 1 & -Z_{\dot{Q}} L / m_3' \\ -M_{\dot{w}} (L I_2'^{-1}) & 1 \end{pmatrix} \qquad B_o = \begin{pmatrix} \frac{Z_{\delta B}' U^2}{m_3' L} & \frac{Z_{\delta S}' U^2}{m_3' L} \\ \frac{M' \delta B U^2}{L^2 I_2'} & \frac{M' \delta S U^2}{L^2 I_2'} \end{pmatrix}$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

3. Dynamic model of the submarine

It holds that the **depth of the vessel** measured **in the inertial reference frame** and the velocity **w** of the submarine along the z-axis of the **body-fixed frame** are related as follows:

$$\begin{aligned}\dot{h} &= w\cos(\theta) - U_o\sin(\theta) \Rightarrow \\ \ddot{h} &= \dot{w}\cos(\theta) - w\sin(\theta)\dot{\theta} - U_o\cos(\theta)\dot{\theta} \Rightarrow \\ \ddot{h} &= \dot{w}\cos(\theta) - wQ\sin(\theta) - U_oQ\cos(\theta)\end{aligned} \quad (4)$$

From the above relation one can compute about the **diving speed** of the vessel:

$$w = (\cos(\theta)^{-1})(\dot{h} + U_o\sin(\theta)) \quad (5)$$

Moreover, from Eq (3) one has:

$$\begin{aligned}\dot{w} &= f_w(w,\theta,Q,t) + B_{o_{11}}u_1 + B_{o_{12}}u_2 \\ \dot{Q} &= f_\theta(w,\theta,Q,t) + B_{o_{21}}u_1 + B_{o_{22}}u_2\end{aligned} \quad (6)$$

Substituting Eq. (5) and the first row of Eq. (6) into Eq. (4) one gets

$$\ddot{h} = [f_w(w,\theta,Q,t) + B_{o_{11}}u_1 + B_{o_{12}}u_2]\cos(\theta) - \frac{(\dot{h}+U_0\sin(\theta))}{\cos(\theta)}Q\sin(\theta) - U_0Q\cos(\theta) \quad (7)$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

3. Dynamic model of the submarine

Next, by denoting:

$$f_w(w,\theta,Q,t) = g_h(h,\dot{h},\theta,\dot{\theta},t)$$
$$f_\theta(w,\theta,Q,t) = g_\theta(h,\dot{h},\theta,\dot{\theta},t)$$

And by substituting this relation in Eq. (7), together with $Q=\dot{\theta}$ one obtains:

$$\ddot{h} = g_h(h,\dot{h},\theta,\dot{\theta},t)cos(\theta) - \frac{(\dot{h}+U_0 sin(\theta))}{cos(\theta)}\dot{\theta}sin(\theta) - U_0\dot{\theta}cos(\theta) + $$
$$+B_{0_{11}}cos(\theta)u_1 + B_{0_{12}}cos(\theta)u_2$$

$$\ddot{\theta} = g_\theta(h,\dot{h},\theta,\dot{\theta},t) + B_{0_{21}}u_1 + B_{0_{22}}u_2$$

Then, by defining the **state vector** $x=[h,\dot{h},\theta,\dot{\theta}]^T$

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_3\end{pmatrix} = \begin{pmatrix} g_b(x,t)cos(x_3) - \frac{x_4+U_0 sin(x_3)}{cos(x_3)}x_4 sin(x_3) - U_0 x_4 cos(x_3) \\ g_\theta(x,t)\end{pmatrix} + \begin{pmatrix} B_{0_{11}} & B_{0_{12}} \\ B_{0_{21}} & B_{0_{22}}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} \quad (8)$$

From Eq. (8) one finally arrives at the **MIMO state-space description** of the submarine

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_3\end{pmatrix} = \begin{pmatrix}f_1(x,t)\\ f_2(x,t)\end{pmatrix} + \begin{pmatrix} g_{11}(x,t) & g_{12}(x,t) \\ g_{21}(x,t) & g_{22}(x,t)\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} \quad (9)$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

4. Differential flatness of the submarine's dynamic model

Next, by denoting the **flat output of the submarine** as:

$$y = [x_1, x_3]^T = [h, \theta]^T$$

it can be proven that the submarine's dynamic model is a **differentially flat** one

This means that all its **state variables** and its **control inputs** can be expressed as **differential functions of the flat output**

From Eq. (9) one gets $x_2 = \dot{x}_1$ and $x_4 = \dot{x}_3$, which means

$$x_2 = [1\ 0]\dot{y} \qquad x_4 = [0\ 1]\dot{y} \qquad (10)$$

Again, from Eq. (9) one gets

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \end{pmatrix}^{-1} \left(\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} - \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \right)$$

which means

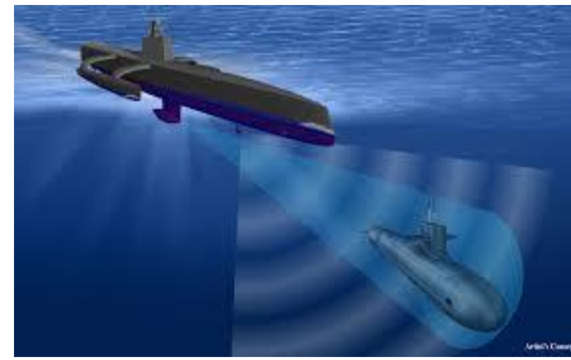
$$u_1 = f_a(y, \dot{y}, \ddot{y}) \qquad u_2 = f_b(y, \dot{y}, \ddot{y}). \qquad (11)$$

Eq. (10) and Eq. (11) confirm that the submarine's model is a differentially flat one.

Example 5: Nonlinear control and state estimation using Lyapunov methods

4. Differential flatness of the submarine’s dynamic model

The differential flatness property of the submarine’s model is important because it means that the vessel’s model can be transformed into the **MIMO linear canonical (Brunovsky) form** through a change of its state variables (diffeomorphism)



By defining the **new state variables of the vessel**

$$y_1 = x_1, \qquad y_2 = \dot{y}_1, \qquad y_3 = x_2, \qquad y_4 = \dot{y}_3$$

and by defining **the transformed control inputs** of the vessel

$$\begin{aligned} v_1 &= f_1(x,t) + g_{11}u_1 + g_{12}u_2 \\ v_2 &= f_2(x,t) + g_{21}u_1 + g_{22}u_2 \end{aligned} \qquad (12)$$

one obtains the **linearized and decoupled state-space model** of the submarine

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad (13)$$

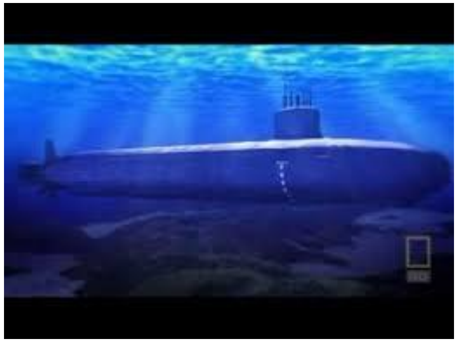
for which the design of a **state-feedback controller** is possible

Example 5: Nonlinear control and state estimation using Lyapunov methods

5. Design of a stabilizing feedback controller for the submarine

For the **transformed state-space model** of the vessel

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (13)$$



It is considered that the complete state vector is measurable

$$y = [h, \dot{h}, \theta, \dot{\theta}]$$

Then, to succeed tracking of the **reference setpoint**

$$y^d = [y_1^d, y_2^d, y_3^d, y_4^d]^T = [x_1^d, \dot{x}_1^d, x_2^d, \dot{x}_2^d]^T$$

the **feedback control inputs** should be chosen as

$$\begin{aligned} v_1 &= \ddot{y}_1^d - k_d^1(\dot{y}_1 - \dot{y}_1^d) - k_p^1(y_1 - y_1^d) \\ v_2 &= \ddot{y}_3^d - k_d^2(\dot{y}_3 - \dot{y}_3^d) - k_p^2(y_3 - y_3^d) \end{aligned} \quad (14)$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

5. Design of a stabilizing feedback controller for the submarine

By substituting Eq. (14) Into Eq. (13) one obtains the **tracking error dynamics** for the submarine

$$\ddot{e}_1 + k_d^1 \dot{e}_1 + k_p^1 e_1 = 0 \qquad \ddot{e}_2 + k_d^2 \dot{e}_2 + k_p^2 e_2 = 0 \qquad (15)$$

where the tracking error is defined as $e_1 = y_1 - y_1^d, \quad e_2 = y_3 - y_3^d$

By selecting the feedback control gains $k_p^i, k_d^i \quad i = 1,2$ so as the **characteristic polynomials**

$$p_1(s) = s^2 + k_d^1 s + k_p^1 \qquad p_2(s) = s^2 + k_d^2 s + k_p^2 \qquad (16)$$

to have roots explicitly in the **left complex semiplane**, it is assured that

$$\lim_{t \to \infty} e_i(t) = 0 \quad i = 1,2$$

Finally, the feedback **control input** that is **actually exerted on the submarine** is .

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} g_{11}(x,t) & g_{12}(x,t) \\ g_{21}(x,t) & g_{22}(x,t) \end{pmatrix}^{-1} [\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}] \qquad (17)$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

6. Differential flatness of the autonomous submarine

- **Differential flatness theory** has been developed as a **global linearization control method** by M. Fliess (Ecole Polytechnique, France) and co-researchers.

- A dynamical system can be written in the ODE form $S_i(w, \dot{w}, \ddot{w}, \dots, w^{(i)}), \quad i = 1,2,\dots,q$

  where $w^{(i)}$ stands for the i-th derivative of either a state vector element or of a control input

- The system is said to be **differentially flat** with respect to the **flat output**

$$y_i = \varphi(w, \dot{w}, \ddot{w}, \dots, w^{(a)}), \qquad i = 1, \dots, m \qquad \text{where} \qquad y = (y_1, y_2, \dots, y_m)$$

if the following two conditions are satisfied

(i) There does not exist any differential relation of the form

$$R(y, \dot{y}, \ddot{y}, \dots, y^{(\beta)}) = 0$$

which means that **the flat output and its derivatives are linearly independent**

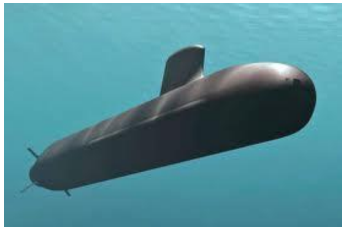
(ii) All system variables are **functions of the flat output and its derivatives**

$$w^{(i)} = \psi(y, \dot{y}, \ddot{y}, \dots, y^{(\gamma_i)})$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

6. Differential flatness of the autonomous submarn

The proposed adaptive control method is based on the **transformation** of the vessel's model into the **linear canonical form**, and this transformation is achieved by exploiting the system's differential flatness properties



- **All single input vessel models** are differentially flat and can be transformed into the linear canonical form

One has to define also which are the **MIMO vessel models** which are differentially flat.

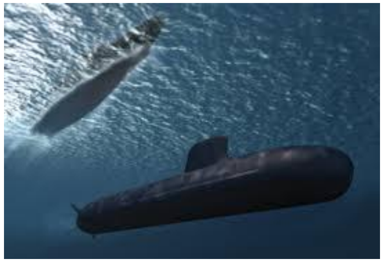
- Differential flatness holds for **MIMO vessel models** that admit **static feedback linearization** and which can be transformed into the linear canonical form through a change of variables (diffeomorphism) and feedback of the state vector. This **is the case of the submarine's model**

- Differential flatness holds for **MIMO vessel models** that admit **dynamic feedback linearization**, This **is the case of underactuated vessel models** In the latter case the state vector of the system is extended by considering as additional flat outputs some of the control inputs and their derivatives

- Finally, a more rare case is the so-called **Liouvillian systems**. These are systems for which differential flatness properties hold for part of their state vector while the non-flat state variables can be obtained by integration of the elements of the flat subsystem.

Example 5: Nonlinear control and state estimation using Lyapunov methods

7. Design of an adaptive controller for the submarine's model



For the **differentially flat MIMO model of the submarine** one has the dynamics

$$\ddot{x}_1 = f_1(x,t) + g_1(x,t)u + \tilde{d}_1$$
$$\ddot{x}_3 = f_2(x,t) + g_2(x,t)u + \tilde{d}_2$$

The following **control input is** considered

$$u = \begin{bmatrix} \hat{g}_1(x,t) \\ \hat{g}_2(x,t) \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \ddot{x}_1^d \\ \ddot{x}_3^d \end{bmatrix} - \begin{bmatrix} \hat{f}_1(x,t) \\ \hat{f}_2(x,t) \end{bmatrix} - \begin{bmatrix} K_1^T \\ K_2^T \end{bmatrix} e + \begin{bmatrix} u_{c_1} \\ u_{c_2} \end{bmatrix} \right\}$$

where $\hat{f}$ and $\hat{g}$ stand for estimates of the unknown nonlinear terms f and g

These estimates are provided by neurofuzzy approximators or other nonlinear regressors

This results in **tracking error dynamics** of the form

$$\dot{e} = (A - BK^T)e + Bu_c + B\left\{ \begin{bmatrix} f_1(x,t) - \hat{f}_1(x,t) \\ f_2(x,t) - \hat{f}_2(x,t) \end{bmatrix} + \begin{bmatrix} g_1(x,t) - \hat{g}_1(x,t) \\ g_2(x,t) - \hat{g}_2(x,t) \end{bmatrix} \begin{bmatrix} \hat{g}_1(x,t) \\ \hat{g}_2(x,t) \end{bmatrix}^{-1} u + \tilde{d} \right\}$$

where matrices A,B,K are defined as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, K^T = \begin{bmatrix} K_1^1 & K_2^1 & K_3^1 & K_4^1 \\ K_1^2 & K_2^2 & K_3^2 & K_4^2 \end{bmatrix}$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

7. Design of an adaptive controller for the submarine's model

The **nonlinear regressors** (neurofuzzy approximators) **consist** of the **kernel functions** and **weights functions.** Unlike SISO systems, in the case of MIMO dynamics the kernel and weights functions are not represented as vectors but **take the form of matrices**. Thus one has:

$$\hat{f}(x|\theta_f) = \Phi_f(x)\theta_f \quad \text{and} \quad \hat{g}(x|\theta_g) = \Phi_g(x)\theta_g$$

Kernel and weights functions for the **approximation of the nonlinear dynamics *f*:**

$$\Phi_f(x) = \begin{bmatrix} \varphi_f^{1,1}(x) & \varphi_f^{1,2}(x) & \cdots & \varphi_f^{1,N}(x) \\ \varphi_f^{2,1}(x) & \varphi_f^{2,2}(x) & \cdots & \varphi_f^{2,N}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_f^{n,1}(x) & \varphi_f^{n,2}(x) & \cdots & \varphi_f^{n,N}(x) \end{bmatrix} \qquad \theta_f^T = [\theta_f^1 \quad \theta_f^2 \quad \cdots \quad \theta_f^N]$$

Kernel and weights functions for the **approximation of the nonlinear dynamics *g***:

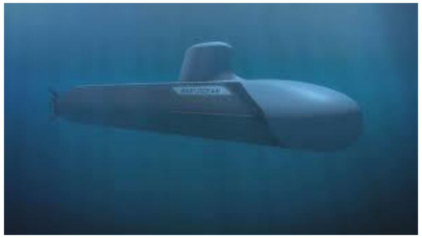
$$\Phi_g(x) = \begin{bmatrix} \varphi_g^{1,1}(x) & \varphi_g^{1,2}(x) & \cdots & \varphi_g^{1,N}(x) \\ \varphi_g^{2,1}(x) & \varphi_g^{2,2}(x) & \cdots & \varphi_g^{2,N}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_g^{n,1}(x) & \varphi_g^{n,2}(x) & \cdots & \varphi_g^{n,N}(x) \end{bmatrix} \qquad \theta_g = \begin{bmatrix} \theta_{g1}^1 & \theta_{g1}^2 & \cdots & \theta_{g1}^p \\ \theta_{g2}^1 & \theta_{g2}^2 & \cdots & \theta_{g2}^p \\ \cdots & \cdots & \cdots & \cdots \\ \theta_{gN}^1 & \theta_{gN}^2 & \cdots & \theta_{gN}^p \end{bmatrix}$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

7. Design of an adaptive controller for the submarine's model

The **weight functions** of the neurofuzzy approximators **are learned** through an **adaptation procedure** that is determined by **Lyapunov stability analysis** for the submarine's model.

The following quadratic **Lyapunov function** is defined:



$$V = \frac{1}{2}e^T P e + \frac{1}{2\gamma_1}\tilde{\theta}_f^T \tilde{\theta}_f + \frac{1}{2\gamma_2} tr[\tilde{\theta}_g^T \tilde{\theta}_g]$$

e: state vector tracking error

$\tilde{\theta}_f = \theta_f - \theta_f^*$: Difference of the weights from the value that succeeds exact estimation of *f*

$\tilde{\theta}_g = \theta_g - \theta_g^*$: Difference of the weights from the value that succeeds exact estimation of *g*

Differentiating one obtains: $\dot{V} = \frac{1}{2}\dot{e}^T P e + \frac{1}{2}e^T P \dot{e} + \frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T \tilde{\theta}_f + \frac{1}{\gamma_2} tr[\dot{\tilde{\theta}}_g^T \tilde{\theta}_g]$

The associated tracking error dynamics is:

$$\dot{e} = (A - BK^T)e + Bu_c + B\{\begin{bmatrix} f_1(x,t) - \hat{f}_1(x,t) \\ f_2(x,t) - \hat{f}_2(x,t) \end{bmatrix} + \begin{bmatrix} g_1(x,t) - \hat{g}_1(x,t) \\ g_2(x,t) - \hat{g}_2(x,t) \end{bmatrix} \begin{bmatrix} \hat{g}_1(x,t) \\ \hat{g}_2(x,t) \end{bmatrix}^{-1} u + \tilde{d}\}$$

The **effect of modelling errors** is denoted by:

$$w = \begin{bmatrix} f_1(x,t) - \hat{f}_1(x,t) \\ f_2(x,t) - \hat{f}_2(x,t) \end{bmatrix} + \begin{bmatrix} g_1(x,t) - \hat{g}_1(x,t) \\ g_2(x,t) - \hat{g}_2(x,t) \end{bmatrix} \begin{bmatrix} \hat{g}_1(x,t) \\ \hat{g}_2(x,t) \end{bmatrix}^{-1} u$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

7. Design of an adaptive controller for the submarine's model

Thus one obtains the following **tracking error dynamics**:

$$\dot{e} = (A - BK^T)e + Bu_c + B(w + \tilde{d})$$

The **first derivative of the Lyapunov function** becomes:

$$\dot{V} = \frac{1}{2}\{e^T(A-BK^T)^T + u_c^T B^T + (w+\tilde{d})^T B^T\}Pe + \frac{1}{2}e^T P\{(A-BK^T)e + Bu_c + B(w+\tilde{d})\}$$
$$+\frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T\tilde{\theta}_f + \frac{1}{\gamma_2}tr[\dot{\tilde{\theta}}_g^T\tilde{\theta}_g]$$

and after intermediate terms substitution one obtains:

$$\dot{V} = \frac{1}{2}e^T\{(A-BK^T)^T P + P(A-BK^T)\}e + \frac{1}{2}2e^T PBu_c + \frac{1}{2}2B^T Pe(w+\tilde{d})$$
$$+\frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T\tilde{\theta}_f + \frac{1}{\gamma_2}tr[\dot{\tilde{\theta}}_g^T\tilde{\theta}_g]$$

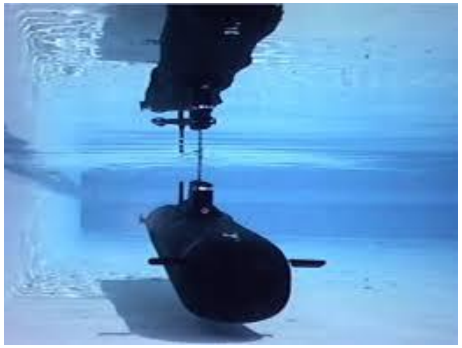
Assumption 1: the positive definite and symmetric matrix P is chosen as solution of the Riccati equation:

$$(A-BK^T)^T P + P(A-BK^T) - PB(\frac{2}{r} - \frac{1}{\rho^2})B^T P + Q = 0$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

7. Design of an adaptive controller for the submarine's model

Using as supervisory control input $u_c = -\frac{1}{r}B^T P e$ one obtains:



$$\dot{V} = \frac{1}{2}e^T\{-Q + PB(\frac{2}{r} - \frac{1}{\rho^2})B^T P\}e + e^T PB\{-\frac{1}{r}B^T Pe\} + B^T P(w + \tilde{d}) +$$
$$+\frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T\tilde{\theta}_f + \frac{1}{\gamma_2}tr[\dot{\tilde{\theta}}_g^T\tilde{\theta}_g]$$

which can be written in the form:

$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) + \frac{1}{\gamma_1}\dot{\tilde{\theta}}_f^T\tilde{\theta}_f + \frac{1}{\gamma_2}tr[\dot{\tilde{\theta}}_g^T\tilde{\theta}_g]$$

Next, substituting: $\dot{\tilde{\theta}}_f = \dot{\theta}_f - \dot{\theta}_f^* = \dot{\theta}_f$ and $\dot{\tilde{\theta}}_g = \dot{\theta}_g - \dot{\theta}_g^* = \dot{\theta}_g$

i.e: $\dot{\theta}_f = -\gamma_1\Phi(x)^T B^T Pe$ and $\dot{\theta}_g = -\gamma_2\Phi(x)^T B^T Peu^T$

the following form of the **derivative of the Lyapunov function** is obtained:

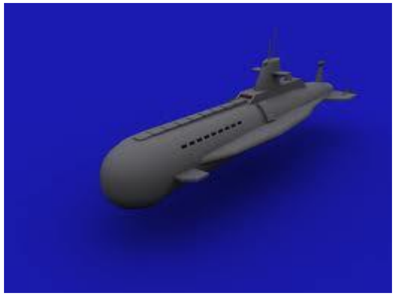
$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) +$$
$$+\frac{1}{\gamma_1}(-\gamma_1)e^T PB\Phi(x)(\theta_f - \theta_f^*) + \frac{1}{\gamma_2}(-\gamma_2)tr[ue^T PB\Phi(x)(\theta_g - \theta_g^*)]$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

7. Design of an adaptive controller for the submarine's model

Taking into account that $u \in R^{2\times 1}$ and $e^T PB(\hat{g}(x|\theta_g) - \hat{g}(x|\theta_g^*)) \in R^{1\times 2}$

the following form is obtained for the **Lyapunov function derivative** :



$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) +$$
$$+\frac{1}{\gamma_1}(-\gamma_1)e^T PB\Phi(x)(\theta_f - \theta_f^*) + \frac{1}{\gamma_2}(-\gamma_2)tr[e^T PB(\hat{g}(x|\theta_g) - \hat{g}(x|\theta_g^*))u]$$

and since $e^T PB(\hat{g}(x|\theta_g) - \hat{g}(x|\theta_g^*))u \in R^{1\times 1}$ it holds that

$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) +$$
$$+\frac{1}{\gamma_1}(-\gamma_1)e^T PB\Phi(x)(\theta_f - \theta_f^*) + \frac{1}{\gamma_2}(-\gamma_2)e^T PB(\hat{g}(x|\theta_g) - \hat{g}(x|\theta_g^*))u$$

Using the following description for the model approximation error:

$$w_a = [\hat{f}(x|\theta_f^*) - \hat{f}(x|\theta_f)] + [\hat{g}(x|\theta_f^*) - \hat{g}(x|\theta_f)]u$$

the equation of the Lyapunov function derivative becomes:

$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PBB^T Pe + e^T PB(w + \tilde{d}) + e^T PBw_a$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

7. Design of an adaptive controller for the submarine's model

and denoting the disturbances and modelling error terms as: $w_1 = w + \tilde{d} + w_a$

one has: $$\dot{V} = -\frac{1}{2}e^TQe - \frac{1}{2\rho^2}e^TPBB^Te + e^TPBw_1$$

or: $$\dot{V} = -\frac{1}{2}e^TQe - \frac{1}{2\rho^2}e^TPBB^Te + \frac{1}{2}e^TPBw_1 + \frac{1}{2}w_1^TB^TPe$$

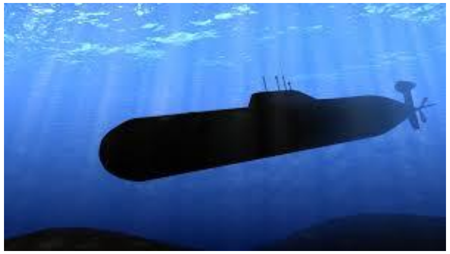
Next **the following inequality** is used:

Lemma: It holds that $$\frac{1}{2}e^TPw_1 + \frac{1}{2}w_1^TB^TPe - \frac{1}{2\rho^2}e^TPBB^TPe \leq \frac{1}{2}\rho^2 w_1^T w_1 \quad (19)$$

Proof:

The binomial $(\rho a - \frac{1}{\rho}b)^2 \geq 0$ is considered. Expanding the left part of the above inequality one gets

$$\rho^2a^2 + \frac{1}{\rho^2}b^2 - 2ab \geq 0 \Rightarrow \frac{1}{2}\rho^2a^2 + \frac{1}{2\rho^2}b^2 - ab \geq 0 \Rightarrow$$
$$ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2a^2 \Rightarrow \frac{1}{2}ab + \frac{1}{2}ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2a^2$$



By substituting $a = w_1$ and $b = \bar{e}^TP_2B$ one gets

$$\frac{1}{2}w_1^TB^TP_2\bar{e} + \frac{1}{2}\bar{e}^TP_2Bw_1 - \frac{1}{2\rho^2}\bar{e}^TP_2BB^TP_2\bar{e} \leq \frac{1}{2}\rho^2w_1^Tw_1$$

Example 5: Nonlinear control and state estimation using Lyapunov methods

7. Design of an adaptive controller for the submarine's model

By substituting Eq. (19) into the relation of the derivative of the Lyapunov function gives:

$$\dot{V} \leq -\frac{1}{2}e^T Q e + \frac{1}{2}\rho^2 w_1^T w_1 \qquad (20)$$

This is the **H-infinity tracking performance criterion** which means that for bounded disturbance and modelling error the control law results in very small bounded tracking error:

It is noted that, by choosing the **attenuation coefficient ρ** to be sufficiently small, the right part of Eq. (20) can be always made to be upper bounded by zero.

In such a case the **asymptotic stability condition** is clear to hold..

The minimum value of ρ for which a solution of the Riccati Eq. (18) exists, is the one that provides the control loop with maximum robustness.

Moreover, if $\int_0^{\infty} ||w_1||^2 dt \leq M_w$ one has the following integral:

$$\int_0^T \dot{V}(t)dt \leq -\frac{1}{2}\int_0^T ||e(t)||^2 dt + \frac{1}{2}\rho^2 \int_0^T ||w_1||^2 dt \Rightarrow 2V(T) + \int_0^T ||e(t)||_Q^2 dt \leq 2V(0) + \rho^2 \int_0^T ||w_1||^2 dt$$

which means that: $\int_0^{\infty} ||e||_Q^2 dt \leq 2V(0) + \rho^2 M_w$ and from **Barbalat's Lemma** one has that

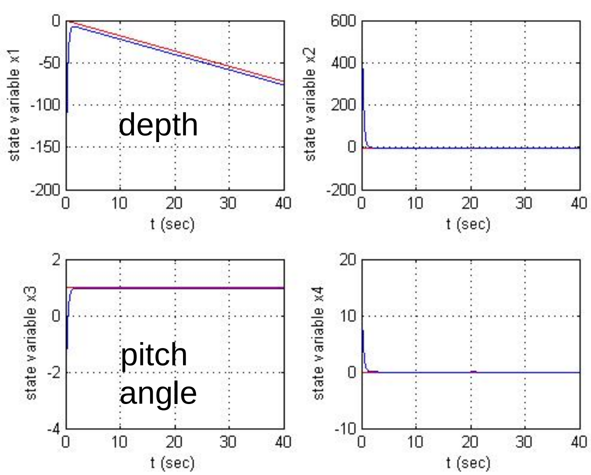
which confirms again that the tracking error vanishes

Example 5: Nonlinear control and state estimation using Lyapunov methods

8. Simulation tests

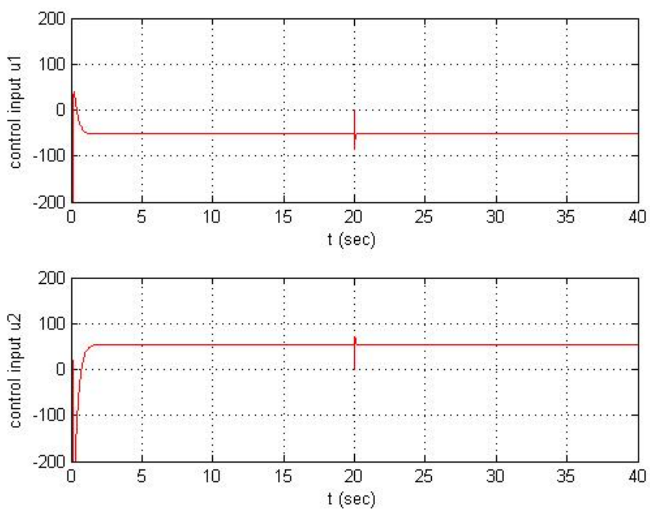
• In the simulation tests, the **dynamic model of the submarine** was considered to be **completely unknown** and was identified in real-time by the previously analyzed nonlinear regressors (neurofuzzy approximators)

• The **estimated unknown dynamics** of the system **was used in the computation of the control inputs** (generated by the electric actuators of the hydroplanes) which were finally exerted on the submarine's model.



state variables $x_i,\ i = 1, \cdots, 4$

Setpoint 1:

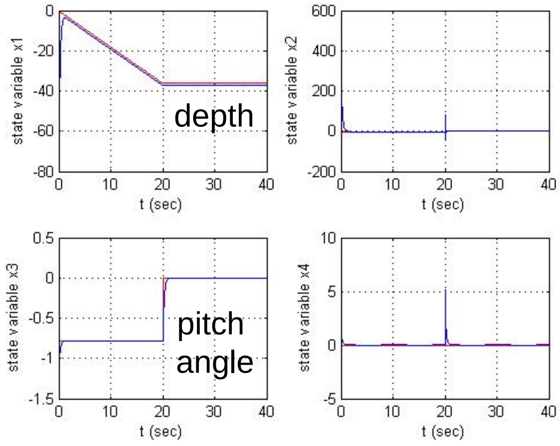


Variations of the control inputs

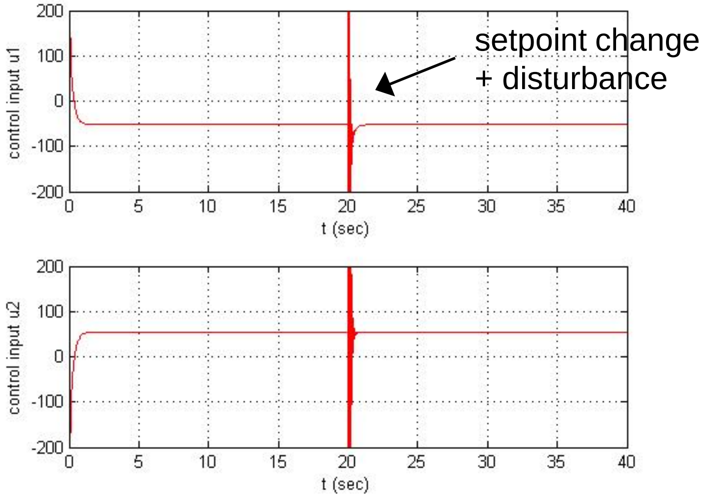
Example 5: Nonlinear control and state estimation using Lyapunov methods

8. Simulation tests

Setpoint 2:

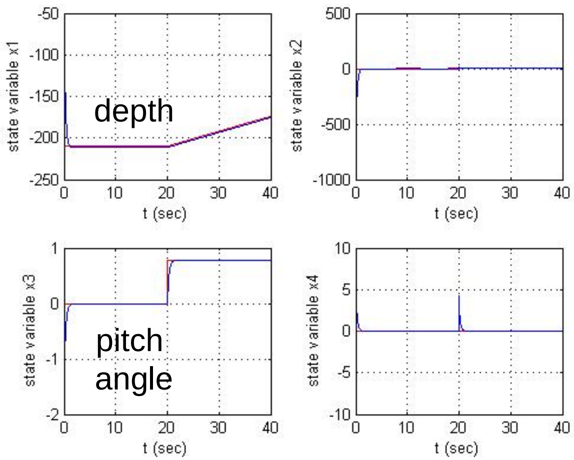


state variables x_i, $i = 1, \cdots, 4$

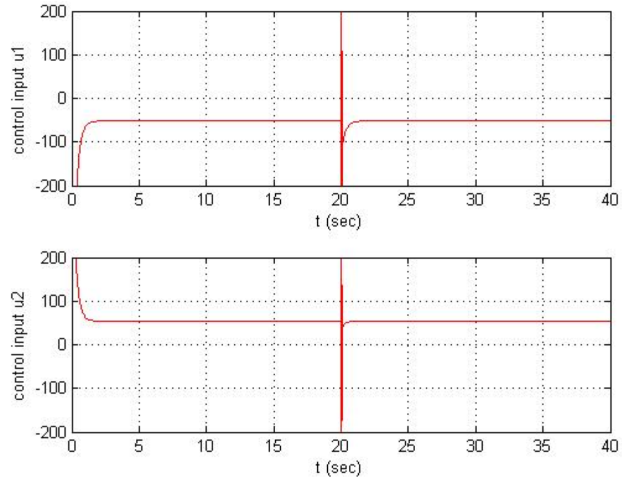


Variations of the control inputs

Setpoint 3:



state variables x_i, $i = 1, \cdots, 4$



Variations of the control inputs

Example 5: Nonlinear control and state estimation using Lyapunov methods

9. Conclusions

• By exploiting the **differential flatness** properties of the **MIMO nonlinear model of the submarine** the system was transformed into the **linear canonical (Brunovsky) form.** For the latter description the design of a feedback controller was possible.

• Moreover, to cope with **unknown nonlinear terms** appearing in the new control inputs of the transformed state-space description of the submarine, the use of nonlinear regressors (neurofuzzy approximators) has been proposed..

• These estimators were online trained to identify the unknown dynamics of the system and the associated learning procedure was determined by the requirement the derivative of the system's Lyapunov function to be a negative one.

• Through **Lyapunov stability analysis** it was proven that the closed loop satisfies the **H-infinity tracking performance criterion**, and this assures improved robustness against model uncertainties and external perturbations.

• The computation of the control input required the **solution of an algebraic Riccati equation**. Suitable selection of the attenuation coefficient ρ in this equation assures asymptotic stability and provides maximum robustness.

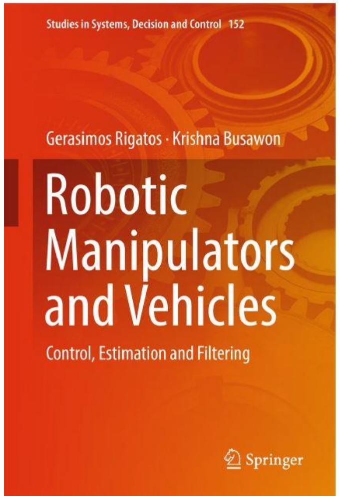
• The proposed flatness-based adaptive fuzzy control method is generic and **can be applied to a wide class of vessels**, such as surface vessels or AUVs.

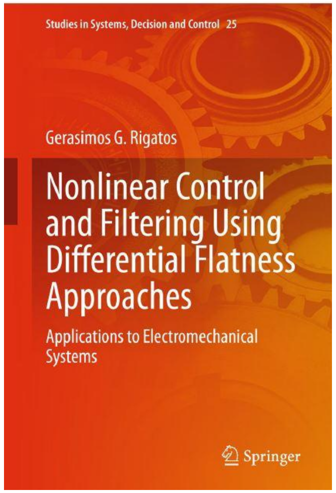
V. Final conclusions

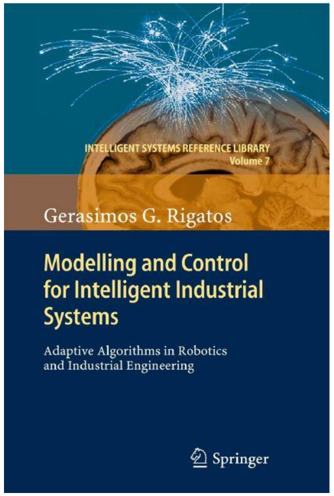
● Methods for nonlinear control and state estimation for autonomous navigation in USVs and AUVs have been developed

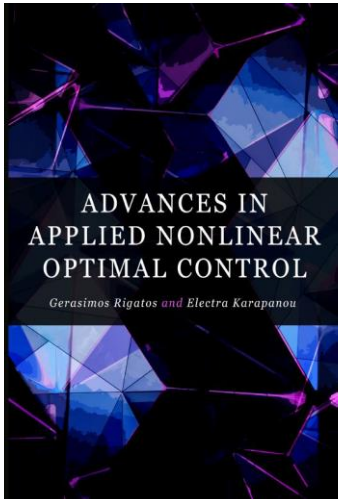
● The main approaches for nonlinear control have been: (i) **control with global linearization** method (ii) **control with approximate (asymptotic) linearization** methods (iii) **control with Lyapunov theory methods (adaptive control)** in case that the dynamic or kinematic model of the USVs and AUVs is unknown

● The main approaches for nonlinear state estimation are: (i) nonlinear state estimation with methods of global linearization (ii) nonlinear state estimation with methods of approximate (asymptotic) linearization









grigat@ieee.org